Box 2
Defining Benchmarks: Introducing Neutral Expectations to the Optimization Process

The process to optimize the strategic selection of assets for the foreign reserve portfolios is based on the Black-Litterman model, which consists of two phases to determine the expected returns on assets. The first involves finding the equilibrium returns implicit in market capitalization of the assets used in the optimization process (fixed-income bond indices) and the estimated covariance matrix, according to a CAPM model.[1] The second phase allows for incorporating expectations of return on these assets and obtaining a weighted average between the two sets of returns (equilibrium and expectations), in addition to including uncertainty about the expectations and the respective modification of the covariance matrix.

Within the framework of this methodology neutral expectations were incorporated into the process of optimizing the foreign reserve portfolios. These neutral expectations are based on information contained in the relevant financial markets (fixed income); i.e., they are the return expectations implicit in the asset prices at the time of optimization. The main reason for including them was the possible mismatch between the expected equilibrium returns that represent long-term performance, and the asset returns and short-term movements (more aligned with the investment horizon of the portfolios) that occur in the market. This allows the latest market data to influence the outcome of optimization.

1. The Black-Litterman Model

As mentioned, the Black-Litterman model has two phases. In the first, it is assumed the market portfolio is efficient (it is on the efficient frontier) and, therefore, knowing the covariance matrix and the market weights, by observing capitalization of the assets, it is possible to determine their equilibrium returns. These can be used for a new optimization that includes the investor's constraints (e.g., maximum probability of loss) when the investor has no expectations. This model assigns higher returns to riskier assets and lower returns to less risky assets, which is the long-term performance expected in an efficient market with rational agents.

The second Black-Litterman phase consists of defining and integrating the expectations of investors' returns with the equilibrium returns of the first phase. To start out, the expectations and their uncertainty are defined,[2] and then a new distribution of modified returns that includes the expectations is obtained.

The following expression shows the second phase of the Black-Litterman model.

$$N(\mu_{eq},\Sigma_{eq})+N(\mu_{exp},\Sigma_{exp}) \xrightarrow{Bayes} N(\mu_{mod},\Sigma_{mod}),$$

where the first term refers to the distribution with equilibrium returns (the first Black-Litterman phase), the second contains the information on the distribution of expectations, and the term on the right is the modified distribution obtained by applying Bayes' theorem. Finally, with the modified parameters, a new optimization is obtained, including the restrictions applicable to the investor. Diagram B2.1 shows the Black-Litterman model in summarized form.

2. Methodology to introduce Market Return Expectations in the optimization of the Foreign Reserves Portfolio

To introduce expectations into the optimization process (the second Black-Litterman phase), it was decided that neutral expectations, implicit in the markets, would be used since they would not introduce subjective biases.

As mentioned, the second Black-Litterman phase requires specifying two sets of parameters: the expected returns and the uncertainty about these expectations. The expected returns are calculated from the implied future spot interest rate curve or forward curve, which is obtained by means of non-arbitrage conditions with the information from the

1 (Box 2-1) Capital asset pricing model (CAPM).

2 (Box 2-2) It should be noted that it is not necessary to define expectations for all assets. So, for example, it is possible to introduce expectations for a single asset.

Diagram B2.1
The Two Steps of the Black-Litterman Model

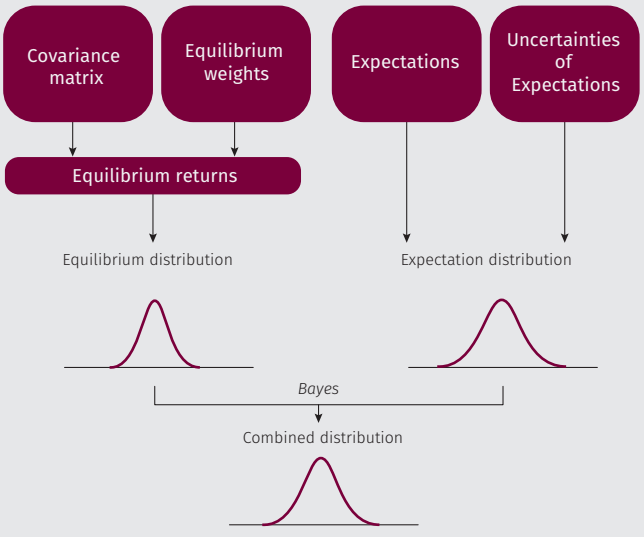


Diagram B2.1
The Two Steps of the Black-Litterman Model

interest rate curve (spot curve) and allows for calculating the components by price change (using the movement in rates between the current and future curves) and by interest (using the average of rates between the curves). To estimate the uncertainty of these forecasts, simulations of the interest rate curve are performed based on the distributions drawn from the interest rate options market. [3] Once the asset returns are calculated in each simulation, their standard deviations are calculated and used as the uncertainty parameter of the expected returns (Diagram B2.2).

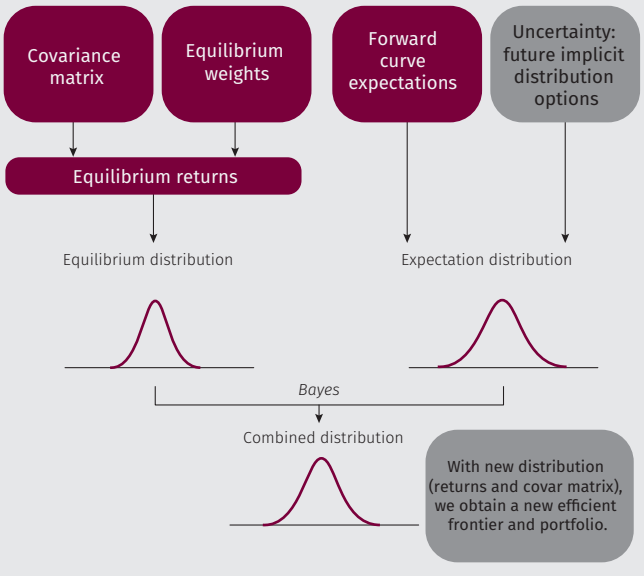
The steps required to find the two sets of parameters needed in the second phase of the Black-Litterman model are described below.

a. Calculating neutral (market) return expectations. Applying the non-arbitrage principle makes it possible to determine the fair rate between two future terms given the spot rates at those terms. The forward curve, which is interpreted as the implicit future spot curve in the horizon equivalent to the shorter term, is obtained by taking the shorter fixed term and varying the second. With these values, it is possible to calculate the vector of return expectations (μexp), which includes the effect of interest rate movements and the expected portfolio causation.

b. Calculating the uncertainties of return expectations. The following steps are required to estimate the uncertainties in return expectations:

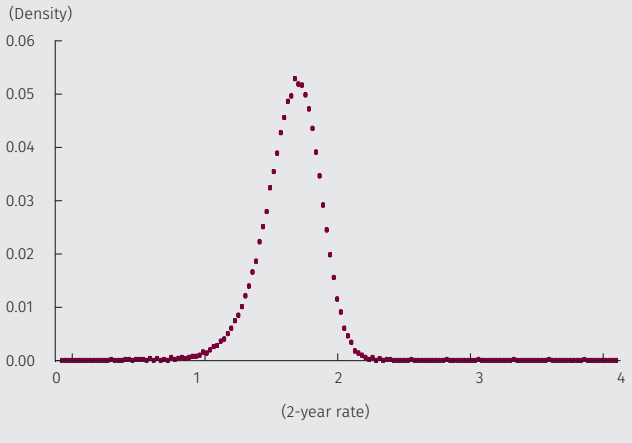
   i. Extraction of implied distributions in the options market. Ever since the work of Breeden-Litzenberg (1978), it has been known that the implied future distribution of an asset is proportional to the curvature of the option price as a function of the strike. Therefore, having a series of option prices for different strikes allows for estimating the future risk-neutral distribution. An example of the distribution found following this procedure is shown in Graph B2.1.

Diagram B2.2
The Two Phases of the Black-Litterman Model Applied to the International Reserves Optimization Exercise



Source: Adapted from "A Step-By-Step Guide to the Black-Litterman Model" by Idzorek (2005).

Graph B2.1
Future Probability Density 2-year Rate



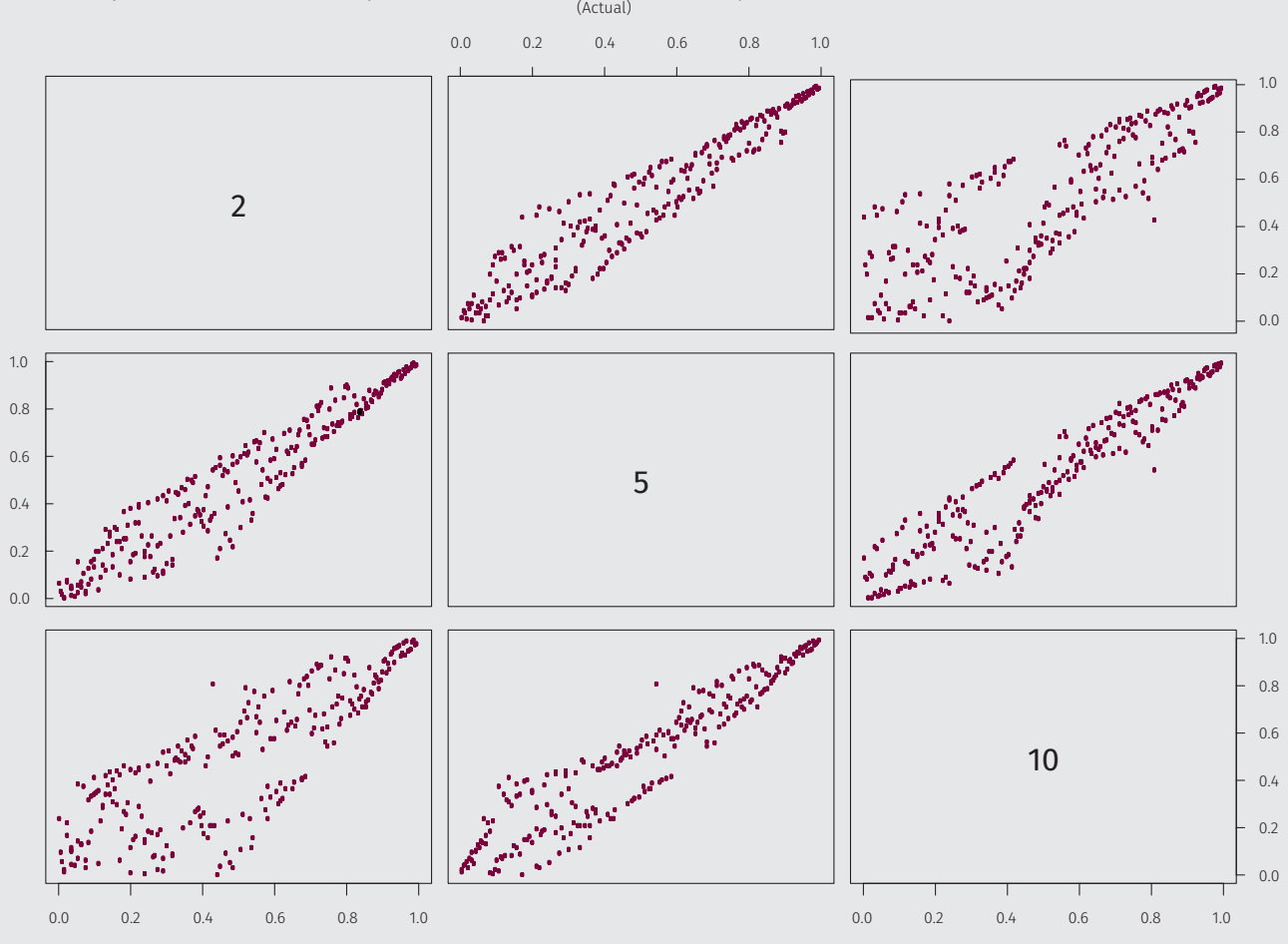
Source: Calculations by *Banco de la República*.

3 (Box 2-3) Since the most developed and liquid markets for interest rate derivatives are for U.S. rates, the forecasts are made on this market.

ii. Copula estimation between different interest rates in the U.S. government bond market. Historical data are taken, and a copula is fitted to the data (Graph B2.2). The copula is a function that generalizes the concept of correlation and, according to Sklar's theorem, exists under conditions that are not too restrictive on the behavior of the distributions of the associated random variables. This step is necessary since interest rates at different maturities in a single currency and of the same credit rating are highly correlated and simulating them independently is not recommended.
iii. Simulation of future interest rates and construction of the future spot curve. Simulations of interest rates are done with the copula of numeral *ii* and the distributions found in numeral *i*. Once the interest rates have been found, the term structure necessary to appraise the bonds is constructed.
iv. Valuation of assets with the new curves and estimation of returns. With the new curves, the returns for the price change component and for the causation component are found and added together.
v. Calculation of the standard deviations of the returns that define the covariance matrix used as an estimator of the uncertainty of expectations. (Σ_{exp}).

The neutral market expectations and their uncertainties are obtained with this procedure and are used to find the new parameters that feed the optimization exercise and the definition of the benchmark index: the vector of expected returns and the variance and covariance matrix of the assets. Using this model, one expects estimates of returns closer to observed values and portfolios more consistent with current market conditions.

Graph B2.2
(Actual)
Relationship between the cumulative probabilities [a/] for the two, five and 10-year rates at the end of the month since 2000.



a/ Cumulative probability distribution functions (pseudo-probabilities) are used since these are inputs for the copula and not the values of the interest rates.
Source: Calculations by *Banco de la República*.