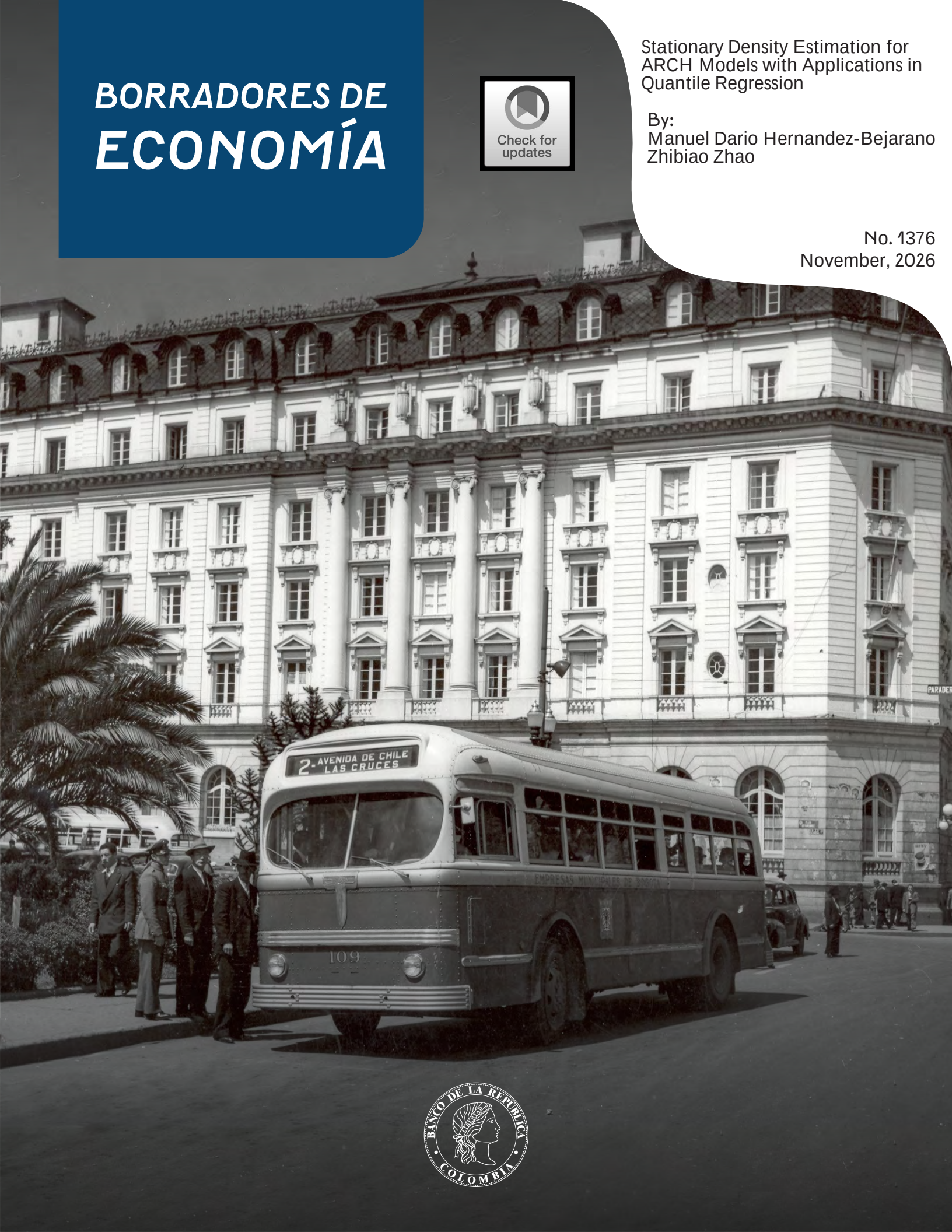
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Stationary Density Estimation for ARCH Models with Applications in Quantile Regression

By:
Manuel Dario Hernandez-Bejarano
Zhibiao Zhao

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Stationary Density Estimation for ARCH Models with Applications in Quantile Regression*

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Manuel Dario Hernandez-Bejarano[†‡]
Zhibiao Zhao[§]

Abstract

This document studies a more efficient density estimation for the stationary density of nonparametric autoregressive conditional heteroscedastic models. These models are important tools in analyzing time series, specifically in economic and financial applications where the goal is to model and understand the volatility of the statistical data. We demonstrate that, in the presence of an unknown nonparametric variance structure, we can establish the $\sqrt{n}$ convergence rate of the proposed density estimator, improving on the widely used nonparametric kernel density estimator, whose convergence rate is inferior. A numerical study confirms the results. The density estimator is applied to the S&P 500 Index data. Finally, we showcase a practical implementation of the proposed density estimator in quantile regression. Specifically, we propose to construct a more accurate estimate of the limiting variance of the estimated coefficients in a quantile regression model with errors following a nonparametric autoregressive conditional heteroscedasticity structure. Simulation studies show that using the new density estimator leads to a more accurate estimation of the asymptotic variance compared to the results obtained by using the classical density estimator. To illustrate the application of this methodology in estimating the asymptotic variance, we apply it to the monthly inflation rate of the United States.

JEL Codes: C12; C14.

Keywords: Density estimation; Functional convergence; Nonparametric kernel density estimator; Nonparametric regression; ARCH models.

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[†]Corresponding author. E-mail: mhernabe@banrep.gov.co

[‡]Research and Econometrics unit, Banco de la República de Colombia, Bogotá, Colombia.

[§]Department of Statistics, The Pennsylvania State University, University Park, USA. E-mail: zuz13@psu.edu

Estimación de densidad estacionaria para modelos ARCH con aplicaciones en regresión cuantílica*

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Manuel Dario Hernandez-Bejarano†‡
Zhibiao Zhao§

Resumen

Este documento estudia una estimación de la densidad más eficiente para la densidad estacionaria de modelos heterocedásticos condicionales autorregresivos no paramétricos. Estos modelos son herramientas importantes para el análisis de series temporales, especialmente en aplicaciones económicas y financieras, donde el objetivo es modelar y comprender la volatilidad de los datos. Demostramos que, en presencia de una estructura de varianza no paramétrica desconocida, podemos establecer la tasa de convergencia $\sqrt{n}$ del estimador de densidad propuesto, mejorando la tasa de convergencia del estimador de densidad kernel no paramétrico ampliamente utilizado, que es inferior. Un estudio numérico confirma los resultados. El estimador de densidad se aplica a los datos del índice S&P 500. Finalmente, presentamos una implementación práctica del estimador de densidad propuesto en el contexto de regresión cuantílica. Específicamente, proponemos construir una estimación más precisa de la varianza límite de los coeficientes estimados en un modelo de regresión de cuantiles con errores que siguen una estructura de heterocedasticidad condicional autorregresiva no paramétrica. Los estudios de simulación demuestran que el uso del nuevo estimador de densidad permite una estimación más precisa de la varianza asintótica que la obtenida con el estimador de densidad clásico. Para ilustrar la aplicación de esta metodología a la estimación de la varianza asintótica, aplicamos el enfoque propuesto a la inflación mensual de Estados Unidos.

Códigos JEL: C12; C14.

Palabras Clave: Estimación de densidad; Convergencia funcional; Estimador de densidad kernel no paramétrico; Regresión no paramétrica; Modelos ARCH.

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†Autor correspondiente. E-mail: mhernabe@banrep.gov.co

‡Unidad de Investigaciones y Econometría, Banco de la República de Colombia, Bogotá, Colombia.

§Departamento de Estadística, Universidad Estatal de Pensilvania, University Park, EE. UU. E-mail: zuz13@psu.edu

1 Introduction

The autoregressive conditional heteroscedastic (ARCH) models proposed by Engle (1982) are helpful tools for analyzing the variance of time series data. The fundamental idea of ARCH models is that the conditional variances are non-constant but change over time. Thus, the application of these models is widespread in situations where the goal is to analyze and forecast volatility since this volatility appears to change over time and exhibit clustering, which are periods of swings interspersed with periods of relative calm [Bollerslev et al. (1994)]. For instance, in finance, one objective is to model the asset returns volatility, as this volatility is considered a measure of risk, and investors want a premium for investing in risky assets [Teräsvirta and Zhao (2011)]. The observations of these asset returns appear to be uncorrelated or nearly uncorrelated, but the data can contain higher-order dependence. Thus, to parameterize this type of dependence, the ARCH model is the most popular model used; see Engle (2001) for examples of applications.

Let Z_i be a real-valued stationary time series of interest, $1 \leq i \leq n$, from the nonparametric autoregressive conditional heteroscedasticity (NARCH) model of the form

$$Z_i = \sigma\left(Z_{i-1}\right) e_i \quad \text{with} \quad e_i \overset{\text{i.i.d.}}{\sim} N(0,1), \tag{1}$$

where $\sigma(\cdot) > 0$ is the unknown conditional variance (or volatility) function, and e_i are i.i.d. error terms which are independent of the past series values Z_{i-1}, Z_{i-2}, ... The nonparametric model (1) does not impose a specific structure on $\sigma(\cdot)$, which lets the data speak for themselves and, in some cases, avoid mis-specification. Also, the normality assumption of the error term is one of the most common assumptions in nonlinear regression, time series, and discrete samplings from diffusion models, and it can be satisfied for many practical data after proper transformations. This assumption is not as restrictive as imposing a parametric form for $\sigma(\cdot)$, which can be difficult or even unrealistic to know in advance in real data. For example, in financial econometrics applications, for discretely sampled data from stochastic diffusion models driven by a standard Brownian motion, the error term is normally distributed[1]. These discrete models are a kind of application of the model (1).

The model (1) can generate heavy-tailed distributions as *kurtosis*$(Z_i) \geq 3$; see Zhao (2008) for details. Thus, the heavy-tailedness characteristic of this model makes it a helpful candidate in many financial and economic applications, where it is frequently observed that the time series exhibit heavy tails. Due to the so-called *curse of dimensionality*, it is practically infeasible to extend the nonparametric model (1) to orders beyond two. We observe that model (1) is a generalization of the ARCH model of order 1 in Engle (1982), which is of the form $Z_i = \sqrt{\alpha_0 + \alpha_1 Z_{i-1}^2}\, e_i$, where α_0 and α_1 are unknown real parameters. Other extension of model (1) include the ARCH model of order p.

The marginal density function plays an important role in the statistical analysis of time-dependent data, as it contains all distributional information about the data. For example, in applications in economic and financial time series [Silverman (1986); Liao and Stachurski (2015)], the marginal density estimation often provides valuable

[1]For a continuous-time diffusion model: $\mathrm{d}Y_t = \sigma(Y_t)\mathrm{d}B_t$, for $t \geq 0$ and $\{B_t\}_{t>0}$ a standard Brownian motion, we have in practice for discrete observations, sample at discrete interval of length $\Delta > 0$ and $t = i\Delta$, we can approximate this model by: $Y_{(i+1)\Delta} - Y_{i\Delta} = \sigma(Y_{i\Delta})\Delta^{1/2}\Delta^{-1/2}[B_{(i+1)\Delta} - B_{i\Delta}]$. Write $Z_i = Y_{(i+1)\Delta} - Y_{i\Delta}$, $Y_i = Y_{i\Delta}$ and $e_i = \Delta^{-1/2}[B_{(i+1)\Delta} - B_{i\Delta}]$, we have that $Z_i = \sigma(Y_i)\Delta^{1/2}e_i$. Then according to the properties of the Brownian motion, we have that e_i has a standard normal distribution.

information about the shape and tail behavior of the distribution of market indices, asset returns, interest rates, GDP growth, inflation, and so on, which is a real need to capture the large fluctuations of these time series. Nonparametric density estimation has been extensively investigated in the literature; in particular, the nonparametric kernel density estimator, the so-called Parzen–Rosenblatt, is one of the most widely used methods for this problem. The properties of this estimator have been studied by Robinson (1983), Ango-Nze and Portier (1994), Ango-Nze and Doukhan (1998), Doukhan and Louhichi (2001) or Roussas (2000) under various types of mixing conditions, by Bosq (1998) in the context of strong mixing sequences, by Dedecker et al. (2007) for different scenarios of weak dependence, and, by Hansen (2008) for stationary strong mixing multivariate data. Denote by $f_Z(z)$ the density of Z. Given $Z_1, \ldots, Z_n$, the Parzen–Rosenblatt estimator [See Silverman (1986)] of $f_Z(z)$ is

$$\tilde{f}_Z(z) = \frac{1}{nb_n} \sum_{i=1}^{n} K\left(\frac{z - Z_i}{b_n}\right), \tag{2}$$

for kernel function $K(\cdot)$ (an integrable function that integrates to 1) and bandwidth $b_n \to 0$. The estimator presented in equation (2) does not require knowledge of the density form of the data. This feature has made it a widely used choice in various fields, including economics and finance, where the underlying density of interest is often unknown, particularly at the beginning of a study. Under appropriate conditions [See Li and Racine (2007)],

$$\sqrt{nb_n}\left[\tilde{f}_Z(z) - f_Z(z) - \frac{b_n^2 f_Z''(z)}{2}\int_{\mathbb{R}} u^2 K(u)du\right] \Rightarrow N\left(0, f_Z(z)\int_{\mathbb{R}} K^2(u)du\right). \tag{3}$$

That is, the estimator $\tilde{f}_Z(z)$ is $\sqrt{nb_n}$-consistent. This slow convergence rate $\sqrt{nb_n}$ of the classical nonparametric kernel density estimator (2) can be problematic for small sample sizes. In addition, the density estimator in (2) does not depend on the subjacent model of the process in (1), being robust to the model assumptions. However, by not considering that particular structure of the process; hence, this estimator is not the most efficient one.

To overcome the aforementioned slow convergence issue of the estimator in (2), $\sqrt{n}$-consistent marginal density estimates have been proposed by Saavedra and Cao (1999, 2000), Schick and Wefelmeyer (2004) for moving average processes, by Kim and Wu (2007) for nonlinear autoregressive models of order one with constant variance, by Schick and Wefelmeyer (2007) for invertible linear processes, by Zhao (2010) for nonlinear autoregressive conditional heteroscedastic models with parametric noise, in which the structure for the functions of mean and variance is known, by Kim et al. (2015) for nonlinear and homocedastic autoregressive processes with nonparametric noise for which the conditional mean has a parametric specification, by Li and Tu (2016) for semiparametric regression models where the mean function is specified as a parametric form up to unknown parameters and the error density is left unspecified, by Delaigle et al. (2016) for the log-volatility of generalized autoregressive conditionally heteroscedastic GARCH(1,1) processes, and, more recently, by Truquet (2019) for autoregressive time series models for which the conditional mean and variance have a parametric specification.

This document has two main goals. For our first goal, motivated by the fact that nonparametric density estimation is a relevant tool for the analysis of time series, as it contains all the distributional information about the data, we propose an $\sqrt{n}$-consistent density estimate for $f_Z(z)$, the marginal density function of Z_i, by utilizing the specific structure of (1). Instead of imposing parametric form on $\sigma(\cdot)$ as is often used in traditional approaches, we impose the normality assumption on noise e while leaving $\sigma(\cdot)$ completely unspecified. In addition to the faster

convergence rate over the nonparametric kernel density estimator (2), the proposed estimator has several other appealing properties. For example, it is uniformly bounded, and possesses certain smoothness properties which the classical nonparametric kernel density estimator lacks. The second goal is to get a more accurate estimate of the limiting variance of the estimated coefficients in a quantile regression model, whose errors follow the nonparametric ARCH structure in (1). We do this by estimating the density function of the error terms in the model $f_Z(z)$, using the proposed efficient estimation of the marginal density.

The rest of the document is structured as follows. Section 2 presents the main results: $\sqrt{n}$-consistency of our proposed density estimator is shown in Subsection 2.1, and an application of the proposed density estimator to quantile regression with nonparametric autoregressive conditional heteroscedastic errors is described in Subsection 2.2. In Section 3, we examine the performance of our density estimate through a simulation study (see Subsection 3.1) and then apply our method to the S&P 500 Index data (refer to Subsection 3.2). Also, we present both simulation data (outlined in Subsection 3.3) and an empirical analysis (see Subsection 3.4) to observe the proposed methodology's performance in estimating the limiting variance of the estimated coefficients in quantile regression with nonparametric ARCH errors. Finally, proofs are gathered in Section A.

2 Main results

Throughout, denote by $\phi\left(z;\sigma^2\right)$ the normal density of $N\left(0,\sigma^2\right)$ and by $\dot{\phi}\left(z;\sigma^2\right)$ the derivative of $\phi\left(z;\sigma^2\right)$ with respect to σ^2.

2.1 Stationary density estimation for nonparametric ARCH model

We propose an efficient $\sqrt{n}$-consistent density estimate for f_Z by utilizing the specific structure of (1). In model (1), conditioning on Z_{i-1}, we have that Z_i has the density function $\phi\left(z;\sigma^2\left(Z_{i-1}\right)\right)$. Thus, by conditioning argument of density function,

$$f_Z(z) = \mathbb{E}\phi\left(z;\sigma^2\left(Z_{i-1}\right)\right). \tag{4}$$

If $\sigma^2(\cdot)$ were known, using the sample version of (4), we can estimate $f_Z(z)$ by

$$f_Z^*(z) = \frac{1}{n}\sum_{i=1}^{n}\phi\left(z;\sigma^2\left(Z_{i-1}\right)\right). \tag{5}$$

When $\sigma^2(\cdot)$ is unknown, it is necessary to estimate it. Let $\hat{\sigma}^2(\cdot)$ be the nonparametric estimate of $\sigma^2(\cdot)$. In view of (5), we propose estimating the stationary density $f_Z(z)$ by

$$\hat{f}_Z(z) = \frac{1}{n}\sum_{i=1}^{n}\phi\left(z;\hat{\sigma}^2\left(Z_{i-1}\right)\right). \tag{6}$$

For $\hat{\sigma}^2(\cdot)$ in (6), we use the widely used nonparametric kernel smoothing estimator (Nadaraya-Watson kernel-weighted) given by

$$\hat{\sigma}^2\left(z\right) = \frac{\sum_{i=1}^{n} Z_i^2 K\left(\left(z-Z_{i-1}\right)/h_n\right)}{\sum_{i=1}^{n} K\left(\left(z-Z_{i-1}\right)/h_n\right)}, \tag{7}$$

for proper kernel function $K(\cdot)$ and bandwidth $h_n \to 0$. Due to the dependence structure in time series data, the optimal bandwidth for dependent data differs from the bandwidth under independence [Herrmann et al. (1992)]. For dependent data, the optimal bandwidth can be expressed as $h_n = \nu h_n^*$, where h_n^* is the bandwidth calculated as if the data were independent [Ruppert et al. (1995)], and $\nu > 1$ is a variance correction factor due to dependence structure. However, estimating ν is generally challenging [Altman (1990);Wu and Zhao (2007)]. For example, based on simulation studies, Li and Zhao (2019) recommend using $\nu = 2$ for their dependent data structure. Our simulations showed that using a constant $\nu = 2$ and the automatic plug-in bandwidth selector h_n^* of Ruppert et al. (1995) (implemented using `R` function `dpill` in the `R` package `KernSmooth`) resulted in a better performance for our proposed estimator in terms of mean squared error. Therefore, we set $h_n = 2h_n^*$ as our approach for selecting the bandwidth h_n in (7) for the simulations and applications discussed in Section 3.

Now, we establish the interesting result that the estimate of $f_Z(z)$ in (6) can achieve $\sqrt{n}$ parametric convergence rate, despite the use of the nonparametric estimate $\hat{\sigma}^2(\cdot)$ in (7). Additionally to the faster convergence rate over the nonparametric kernel density estimator $\tilde{f}_Z(z)$ in (2), the proposed estimator $\hat{f}_Z(z)$ has other appealing properties, such as being uniformly bounded and possessing smoothness features which the classical nonparametric kernel density estimator $\tilde{f}_Z(z)$ lacks for small sample sizes. We now give some assumptions for establishing the $\sqrt{n}$-consistency of the proposed density estimate of $f_Z(z)$ in (6).

Assumption 1. (i) The kernel function $K(\cdot)$ is symmetric and continuously differentiable, has bounded derivative and bounded support, satisfies $\int_{\mathbb{R}} K(u)du = 1$ and $\int_{\mathbb{R}} uK(u)du = 0$ (implication of symmetry). (ii) h_n satisfies $nh_n^4 \to 0$, $\sqrt{n}h_n/(\log n)^2 \to \infty$. (iii) $\sigma(\cdot)$ is twice continuously differentiable. There exists $\sigma_0 > 0$ such that $\sigma(z) \geq \sigma_0$ for all $z \in \mathbb{R}$.

Assumption 2. (i) $\{e_i\}$ are i.i.d. and $\{Z_i\}$ is causal in the sense that Z_i depends only on $\mathcal{F}_i$, the sigma field generated by $\{e_j\}_{j \leq i}$. (ii) The sequence $\{Z_i\}$ is strictly stationary and strongly mixing with $\alpha-$mixing coefficients $\alpha_k = O(\rho^k)$ for some $\rho \in (0, 1)$. Furthermore, $\mathbb{E}(|Z_i|^s) < \infty$ for some $s > 2$.

Condition (i) in Assumption 1 is typical in nonparametric kernel smoothing literature. We impose boundedness on the support of $K(\cdot)$ for the brevity of proofs; it may be removed at the cost of lengthier proofs. In particular, the Gaussian kernel is allowed because its tails are thin. In nonparametric regression, the usual bandwidth conditions are $h_n \to 0$ and $nh_n \to \infty$, and condition (ii) in Assumption 1 imposes a stronger bandwidth condition in order to suppress the bias effect and establish $\sqrt{n}$ convergence rate of our estimator. Assumption 2(ii) is frequently used to control dependence structures in time series, and it is reasonably weak and fulfilled for many time series models. Also, the exponentially decaying mixing coefficient assumption can be replaced by some polynomial decaying assumption with much lengthier proofs; See Hansen (2008).

The following Theorem describes the asymptotic behavior of our proposed estimator $\hat{f}_Z(z)$.

Theorem 1. *For $\hat{f}_Z(z)$ in (6) with $\hat{\sigma}(\cdot)$ defined in (7), under Assumptions 1 and 2, for any bounded interval $\mathcal{Z}$, the functional Central Limit Theorem (CLT) holds:*

$$\left\{\sqrt{n}\left[\hat{f}_Z(z) - f_Z(z)\right], \quad z \in \mathcal{Z}\right\} \Rightarrow \{G(z), \quad z \in \mathcal{Z}\} \tag{8}$$

for a centered Gaussian process $\{G(z)\}$ *with autocovariance* $\Sigma(z,z') = \mathrm{Cov}\{G(z), G(z')\}$ *given by*

$$\Sigma(z,z') = \gamma_0(z,z') + \sum_{k=1}^{\infty}[\gamma_k(z,z') + \gamma_k(z',z)] \quad \textit{with} \quad \gamma_k(z,z') = \mathbb{E}[\xi_i(z)\xi_{i+k}(z')], \tag{9}$$

where

$$\xi_i(z) = [\phi(z;\sigma^2(Z_{i-1})) - \mathbb{E}\phi(z;\sigma^2(Z_{i-1}))] + \dot{\phi}\left(z;\sigma^2\left(Z_{i-1}\right)\right)\sigma^2(Z_{i-1})(e_i^2-1). \tag{10}$$

Here we give an intuitive explanation for the covariance structure in (9). For time series data $X_1,\ldots,X_n$, the CLT for the sample mean $\bar{X}$ states that, under appropriate dependence conditions, $\sqrt{n}[\bar{X} - \mathbb{E}(X_0)] \Rightarrow N(0,\sigma^2)$, where $\sigma^2 = \mathrm{Cov}(X_0,X_0) + 2\sum_{k=1}^{\infty}\mathrm{Cov}(X_0,X_k)$ is the long-run variance. In (9), $\Sigma(z,z')$ has a similar long-run variance structure. Note that, since Z_{i-1} and e_i are independent (the causal condition in Assumption 2), $\mathbb{E}[\dot{\phi}\left(z;\sigma^2\left(Z_{i-1}\right)\right)\sigma^2(Z_{i-1})(e_i^2-1)] = 0$ and $\mathbb{E}\xi_i(z) = 0$. Therefore, in the absence of dependence, $\gamma_i(z,z') = 0$ for all $i \geq 1$, and $\Sigma(z,z')$ would contain only the first term $\gamma_0(z,z')$.

In (10), we observe that $\xi_i(z)$ has two components $\xi_{i,1}(z) := [\phi(z;\sigma^2(Z_{i-1})) - \mathbb{E}\phi(z;\sigma^2(Z_{i-1}))]$ and $\xi_{i,2}(z) := \dot{\phi}\left(z;\sigma^2\left(Z_{i-1}\right)\right)\sigma^2(Z_{i-1})(e_i^2-1)$. The first component $\xi_{i,1}(z)$ results from the variability introduced by using the sample version (5) to estimate the population version (4), and $f_Z^*(z)$ in (5) admits the same function CLT in Theorem 1 with $\xi_i(z)$ therein being replaced by just the first component $\xi_{i,1}(z)$. As can be seen from the proof of Theorem 1, the second component $\xi_{i,2}(z)$ stems from the variability caused by the estimate $\hat{\sigma}^2(z)$ of $\sigma^2(z)$.

By Theorem 1, for each fixed z, we have that

$$\sqrt{n}[\hat{f}_Z(z) - f_Z(z)] \Rightarrow N\Big(0, \Sigma\left(z,z'\right)\Big). \tag{11}$$

We observe that the proposed estimate $\hat{f}_Z(z)$ in (6) posses a faster convergence rate $\sqrt{n}$ compared to convergence rate $\sqrt{nb_n}$ of $\tilde{f}_Z(z)$ in (2). Intuitively, the estimate $\hat{f}_Z(z)$ makes use of the information from the conditional heteroscedasticity structure which, while being nonparametric, can still provide information to the density through the identity (4), which leads to more efficient density estimation. Below, we highlight some additional appealing characteristics of this new density estimator.

(a) *Uniform boundedness.* Since $\phi\left(z;\sigma^2(z_{i-1})\right)$ is uniformly bounded, $\hat{f}_Z(z)$ has the same uniform bound. In statistical inference, we often require the density function be bounded, thus $\hat{f}_Z(z)$ satisfies this property. However, for $\tilde{f}_Z(z) = (nb_n)^{-1}\sum_{i=1}^{n} K\left\{\left(z - Z_i\right)/b_n\right\}$ in (2), even though $K(\cdot)$ is bounded, there always exist sample paths (with asymptotically vanishing probability though) such that all Z_i 's are close to z so that $\tilde{f}_Z(z) = O\left(1/b_n\right)$ can be arbitrarily large as $b_n \to 0$.

(b) *Smoothness of the limiting process* $\{G(z)\}$ *in Theorem 1.* Since $\xi_i(z)$ in (10) is differentiable, the process $\{G(z)\}$ is differentiable. In general, the limiting process $\{G(z)\}$ is infinitely differentiable. However, the nonparametric kernel density estimator $\tilde{f}_Z(z)$ in (2) exhibits completely different behavior. To see this, let

$$U_n(z) = \frac{\sqrt{nb_n}[\tilde{f}_Z(z) - f_Z(z) - b_n^2 f_Z''(z)\int_{\mathbb{R}} u^2K(u)du/2]}{\sqrt{f_Z(z)\int_{\mathbb{R}} K^2(u)du}}.$$

Under Assumption 1, it is well known that $U_n(z) \Rightarrow N(0,1)$ for each fixed z and that $U_n(z)$ and $U_n(z')$ are asymptotically independent $N(0,1)$ for $|z-z'| > 2Lb_n$, where we assume $K(\cdot)$ has bounded support $[-L, L]$. If we take $|z-z'| = 3Lb_n \to 0$, then $U_n(z) - U_n(z') \Rightarrow N(0,2)$. That is, even when z and z' are arbitrarily close, the distance $U_n(z) - U_n(z')$ is a non-degenerate random variable $N(0,2)$, indicating discontinuity. Roughly speaking, this means that, for large enough n, $U_n(z)$ consists of pure jumps without any continuity.

(c) *Functional CLT.* The undesirable non-smoothness of $\tilde{f}_Z(z)$ discussed in (b) is caused by the lack of tightness, which is why $\tilde{f}_Z(z)$ only has pointwise asymptotic normality and does not admit the type of functional CLT of $\hat{f}_Z(z)$ as in Theorem 1.

To apply Theorem 1 for statistical inference purposes (for instance, confidence interval construction), now we discuss the estimation of the covariance function $\Sigma(z, z')$ in (9) and we will do so by estimating $\gamma_i(z, z')$. A natural estimate of $\gamma_k(z, z')$ is to replace the population version $\mathbb{E}[\xi_i(z)\xi_{i+k}(z')]$ by its sample version:

$$\hat{\gamma}_k(z, z') = \frac{1}{n-k} \sum_{i=1}^{n-k} \hat{\xi}_i(z)\hat{\xi}_{i+k}(z'), \tag{12}$$

where $\hat{\xi}_i(z)$ is an estimate of $\xi_i(z)$ given by

$$\hat{\xi}_i(z) = [\phi(z; \hat{\sigma}^2(Z_{i-1})) - \hat{f}_Z(z)] + \dot{\phi}\left(z; \hat{\sigma}^2\left(Z_{i-1}\right)\right)\left[Z_i^2 - \hat{\sigma}^2(Z_{i-1})\right]. \tag{13}$$

Note that, in (10), $\sigma^2(Z_{i-1})(e_i^2 - 1) = Z_i^2 - \sigma^2(Z_{i-1})$, which leads us to use $Z_i^2 - \hat{\sigma}^2(Z_{i-1})$ in (13).

Proposition 1. *Under the same conditions in Theorem 1, we have uniform consistency*

$$\sup_{z,z' \in \mathcal{Z}} |\hat{\gamma}_k(z, z') - \gamma_k(z, z')| = o_p(1).$$

With the estimate $\hat{\gamma}_k(z, z')$, from (9), we can estimate $\Sigma(z, z')$ by

$$\hat{\Sigma}(z, z') = \hat{\gamma}_0(z, z') + \sum_{k=1}^{L} [\hat{\gamma}_k(z, z') + \hat{\gamma}_k(z', z)], \tag{14}$$

where L is the lag truncation parameter. For long run variance estimation under time series setting, various approaches have been proposed [Hansen (1992), White (1984), Newey and West (1987) and Andrews (1991)], and (14) is a special case with truncated kernel. As discussed in the aforementioned works, the optimal choice for lag truncation parameter $L = Cn^{1/3}$, where C depends on other unknown quantities that can be difficult to estimate. In practice, an ad hoc choice is $L = \lfloor n^{1/3} \rfloor$.

According to the asymptotic distribution in (3), the classical nonparametric kernel estimator $\tilde{f}_Z(z)$ possesses bias. Thus, to conclude this section, we briefly discuss the issue of bias in estimating the density $f_Z(z)$ using $\tilde{f}_Z(z)$ in (2). From Li and Racine (2007), we have that the bias of $\tilde{f}_Z(z)$, at any point z, can be calculated as follows:

$$\text{Bias}\left(\tilde{f}_Z(z)\right) = [1 + o(1)]\frac{b_n^2 f_Z''(z)}{2} \int_{\mathbb{R}} u^2 K(u) du, \tag{15}$$

where the expression $f''_Z(z) = \partial^2 f_Z(z)/\partial z^2$ represents the second-order derivative of density $f_Z(z)$. As explained by Racine (2008), the above equation for bias has the pointwise property, which means it holds at any point z. We can observe the following facts from (15). First, the bias of $\tilde{f}_Z$ depends on the bandwidth b_n. As b_n decreases, the bias falls. Second, this bias is caused by the curvature of the density function, which means that the bias may increase with $f''_Z(z)$, hence is highest in the peaks of the distribution. This is because, at the peaks of the distribution, $\tilde{f}_Z$ tends to smooth the distribution too much, thereby making the density function estimation smoother (less curved) than it actually is (we will observe this behavior through a simulation study in Subsection 3.3). However, as long as the bandwidth $b_n \to 0$ as $n \to \infty$ (bias $\to 0$), then the bias associated with $f''_Z(z)$ will become smaller as the available data increase, and eventually disappear. For further details, please refer to Racine (2008) or Li and Racine (2007).

Now, the bias of the classical nonparametric kernel density estimator in (15), under the model in (1) is described in the following Corollary 1. Additionally, in Subsection 3.3, we present a simulation study that aims to quantify this bias of $\tilde{f}_Z(z)$.

Corollary 1. *For the bias of $\tilde{f}_Z(z)$ in (15), with Gaussian kernel function $K(\cdot)$ and bandwidth $b_n \to 0$, if we consider the nonparametric autoregressive conditional heteroscedasticity (NARCH) model in (1), then the bias of $\tilde{f}_Z(z)$ is of the form of* $\text{Bias}(\tilde{f}_Z(z)) = [1 + o(1)]b_n^2 C_Z(z)$, *where*

$$C_Z(z) = \frac{1}{2}\mathbb{E}\left[\frac{1}{\sigma^3(Z)}\left(\frac{z^2}{\sigma^2(Z)} - 1\right)\phi\left(\frac{z}{\sigma(Z)}; 1\right)\right]. \tag{16}$$

From (16) in Corollary 1, we can observe two particular cases. Firstly, if $z = 0$, then

$$C_Z(0) = -\frac{1}{2\sqrt{2\pi}}\mathbb{E}\left[\frac{1}{\sigma^3(Z)}\right] < 0.$$

Thus, in this scenario, the estimator $\tilde{f}_Z$ will tend to underestimate the true density f_Z, and this issue can be significant when $\sigma(z)$ is relatively small. Secondly, if z is much larger than the typical values of $\sigma(Z)$, then $z^2/\sigma^2(Z) - 1 > 0$ and $C_Z(z) > 0$. For example, if $c^* < \sigma(Z) < C^*$ for some $0 < c^* < C^*$, then $C_Z(z) > 0$ for all $z > C^*$. In particular, if $z \approx \pm\infty$, then $C_Z(z) > 0$. In this scenario, the estimator $\tilde{f}_Z$ will show a tendency to overestimate the true density f_Z for large positive or negative values of z, i.e., overestimate the extreme tails. These features of bias for the classical nonparametric kernel density estimator will help us to explain the performance of this estimator, compared with our proposed estimator $\hat{f}_Z(z)$, in estimating the limiting variance of the estimated coefficients in the quantile regression model with nonparametric ARCH errors in the simulation study in Subsection 3.3.

2.2 Quantile regression with nonparametric ARCH errors

Quantile regression has recently received special attention in studying time series to address important issues in many disciplines, such as economics and finance. This method introduced by Koenker and Bassett (1978) provides a way of estimating the conditional quantile function and, thus, the whole conditional distribution of the variable of interest on the explanatory variables in the regression model. It also broadens the modeling options available for time series analysis. Quantile regression has been applied to diverse statistical models as it is robust against the

presence of outliers and allows extracting helpful information to make further inferences. For a general review of quantile regression, see the monograph by Koenker (2005) or Koenker et al. (2018) which also shows a wide range of additional examples. This section aims to obtain a more accurate estimate of the limiting variance of estimated coefficients in quantile regression with nonparametric ARCH errors. We achieve this by efficiently estimating the marginal density of the error term using the proposed density estimator $\hat{f}_Z(\cdot)$.

Consider observations Y_i, $1 \leq i \leq n$, from the linear regression model with nonparametric ARCH errors

$$Y_i = X_i^T \beta + Z_i \quad \text{with} \quad Z_i \sim \text{ARCH model in (1)}, \tag{17}$$

where $\beta = (\beta_0, \beta_1, \ldots, \beta_{d-1})^T \in \mathbb{R}^{d\times 1}$ is a d-dimensional coefficient vector of the vector of regressors $X_i = (1, X_{1i}, \ldots, X_{(d-1)i}) \in \mathbb{R}^{1\times d}$ in the regression model and the error term Z_i is independent of X_j, $j \leq i$. Also, we assume that the error terms $\{Z_i\}$ have a common continuous distribution function F_Z and stationary density function f_Z. As the error terms follow the model in (1), $\sigma(Z_{i-1}) > 0$ represents the unknown conditional variance function of Z_{i-1}. In practice, X_i may contain lagged values of Y_i and lagged values of other covariates, for example $X_i = Y_{i-1}$.

As an alternative to the least squares regression that employs information from the conditional mean of the dependent variable given a set of regression variables, X_i, quantile regression provides an appealing method, to estimate the model in (17), since it can give a more informative picture of the distribution by specifying different conditional quantile positions $\tau \in (0, 1)$ of the dependent variable given a set of predictors in the regression model [Koenker (2005)]. Thus, we use the quantile regression approach to estimate the parameters β in the model in (17).

In the context of linear regression models with heteroscedastic errors, parametric specification of the conditional variance of errors, $\sigma(\cdot)$, has been extensively studied in the literature. The use of two-step methods is often suggested for the estimation of the model. For instance, in the quantile regression for the linear heteroscedastic model investigated by Koenker and Zhao (1994) and Gutenbrunner and Jurečková (1992), $\sigma(X_i)$ is assumed to be a linear function of the regression variables, $\sigma(X_i) = X_i^T\gamma$, where γ is a k-dimensional vector, which can be estimated by regressing the absolute residuals from initial estimation on X_i. Koenker and Zhao (1996) discuss the case of quantile regression for linear parametric ARCH models, with $\sigma(\epsilon_i) = \epsilon_i^T\gamma$ as a linear function of the absolute residuals $\epsilon_i = (1, |Z_{i-1}|, \ldots, |Z_{i-q}|)$, where γ is a $(q+1)$-dimensional vector, which is computed based on the estimated residuals of the model. Likewise, for an example of the nonparametric specification of the conditional variance of errors, Zhao (2001) studies the median regression models in the presence of heteroscedasticity of unknown form, in the scenario of independent observations, where $\sigma(X_i)$ is an unknown function of the regression variables, which is estimated in a nonparametric way based on the estimated residuals of the model.

Let $F_Z^{-1}(\tau)$ be the τth quantile of the error term Z_i. The conditional quantile functions of Y_i in the model (17) given X_i, are of the form of

$$Q_{Y_i}(\tau \mid X_i) = X_i^T\beta + F_Z^{-1}(\tau) = X_i^T\beta(\tau), \tag{18}$$

where $\beta(\tau) = (\beta_0 + F_Z^{-1}(\tau), \beta_1, \ldots, \beta_{d-1})^T$. In many applications, conditional heteroscedasticity is often modeled based on the residuals with the effect of the location component removed [see, e.g., Engle (1982); Gutenbrunner

and Jurečková (1992); Koenker and Zhao (1994, 1996) among many others]. For instance, in Engle (1982), the estimation of the ARCH regression model is performed in two steps: the first step involves estimating the autoregressive parameters by least squares and computing the residuals, and the second step consists in estimating the parameters of the ARCH component by regressing the squared residuals on the lagged squared residuals. Similarly, Koenker and Zhao (1996) use a two-step procedure to estimate the quantile regression for linear parametric ARCH models. In this section, we use a two-step procedure to estimate the model in (17) and analyze the behavior in the estimation of the limiting variance of $\hat{\beta}(\tau)$ when the proposed estimator $\hat{f}_Z$ is used.

The two-step strategy to estimate model (17) is as follows:

Step 1: Perform the conditional quantile estimation to obtain an estimator of the d-dimensional coefficient vector $\beta(\tau)$. Then, as Koenker and Bassett (1978), for any fixed τ, the τ-th regression quantile estimator of $\beta(\tau)$ is given by

$$\hat{\beta}(\tau) = \underset{b\in\mathbb{R}^d}{\operatorname{argmin}} \sum_{i=1}^{n} \rho_\tau \left(Y_i - X_i^T b\right), \tag{19}$$

where $\rho_\tau(u) = u\,(\tau - I(u<0))$ is τ-th quantile loss, defined for $0<\tau<1$, which simply gives different weights to positive and negative values and $I(\cdot)$ denotes the indicator function. For example, for $\tau = 0.5$, the objective function (19) reduces to the recognized *Least Absolute Deviation* [Pollard (1991)] maximization problem: $\operatorname{argmin}_{b\in\mathbb{R}^d} \sum_{i=1}^{n} \left|Y_i - X_i^T b\right|$. The calculation of the regression quantiles by using standard linear programming techniques is quite efficient. For a thorough description of algorithmic aspects, see Koenker and D'Orey (1987). We use the function `qr` in the `R` package `quantreg` to get the estimates in (19) for different values of τ.

Step 2: Compute the estimated noises:

$$\hat{Z}_i = Y_i - X_i^T \hat{\beta}(\tau), \quad i = 1, \ldots, n., \tag{20}$$

and utilize the resulting $\hat{Z}_i$ values to estimate the conditional variance of the error terms, $\sigma(\cdot)$, by applying the widely used nonparametric kernel smoothing estimate $\hat{\sigma}^2(\cdot)$ given in (7). The quantity $\hat{Z}_i$ is specific to the τ-th quantile. To select the bandwidth h_n of the estimator $\hat{\sigma}^2(\cdot)$, as discussed in Subsection 2.1, use $h_n = 2h_n^*$, where h_n^* is the automatic plug-in bandwidth selector of Ruppert et al. (1995), which is implemented by using the command `dpill` in the `R` package `KernSmooth`.

It is well known [Koenker (2005)] that the limiting variance of $\hat{\beta}(\tau)$ in (19) depends on the reciprocal of the density function of the model error terms, evaluated at the quantile of interest, $1/f_Z(F_Z^{-1}(\tau))$, called the sparsity. Thus, to compute the estimation of the precision of the τ-th quantile regression estimate directly, the unknown quantity $f_Z(F_Z^{-1}(\tau))$ needs to be estimated, which is hard to do adequately. Then, an efficient estimate of the density, $\hat{f}_Z(\cdot)$, will give us a more accurate estimate of this limiting variance to have the possibility to make inference about the estimation of $\hat{\beta}(\tau)$, for example, to construct confidence intervals. Now, based on Koenker et al. (2018), we give some regularity assumptions for establishing the asymptotic behavior of the quantile regression estimator, $\hat{\beta}(\tau)$.

Assumption 3. The density of the error terms f_Z is bounded and continuous, that $0 < f_Z\left(F_Z^{-1}(\tau)\right) < \infty$ for any $0<\tau<1$.

Assumption 4. (i) For each i, $1 \leq i \leq n$, Z_i is independent of the series values Z_{i-2}, X_i, X_{i-1}, ... (ii) Let $\{X_i\}$ be strictly stationary and ergodic process, with $E\left(\|X_t\|^2\right) < \infty$. (iii) Let $\{Z_i\}$ as in assumption 2(ii). (iv) There exist positive definite matrix Ω_0 such that $\lim_{n\to\infty} n^{-1} \sum X_i X_i^\top = \Omega_0$ and $\max_{i=1,\ldots,n} \|X_i\| / \sqrt{n} \to 0$.

Assumption 3 is a common assumption in quantile regression [see Koenker (2005)]. The dependence structure in Assumption 4 offers a very flexible framework to study asymptotic theory. Also, we focus on the model with stationary covariates; hence, Assumption 4(ii) assumes that $\{X_t\}$ is strictly stationary. Assumption 4(iv) is familiar throughout the literature on M-estimators for regression models [see Koenker (2005)]. Then, following the result according to Koenker et al. (2018), under Assumptions 3 and 4, for any fixed $\tau \in (0,1)$, we have that the asymptotic behavior of $\hat{\beta}(\tau)$ in (19), in step 1, is given by

$$\sqrt{n}(\hat{\beta}(\tau) - \beta(\tau)) \Rightarrow N\left(0, \Gamma(\hat{\beta}(\tau))\right), \tag{21}$$

where the asymptotic variance $\Gamma(\hat{\beta}(\tau))$ is of the form of

$$\Gamma(\hat{\beta}(\tau)) = \frac{\tau(1-\tau)}{f_Z^2(F_Z^{-1}(\tau))} \Omega_0^{-1}. \tag{22}$$

From (22), as we mentioned above, the limiting variance $\Gamma(\hat{\beta}(\tau))$ depends on the quantity $f_Z(F_Z^{-1}(\tau))$. However, in practice, both the stationary density, f_Z, and the error terms Z_i are unknown, so we need to estimate these two quantities to have an estimation of $f_Z(F_Z^{-1}(\tau))$. Thus, for any fixed $\tau \in (0,1)$, the procedure to estimate the sparsity is as follows:

(1) Compute $\hat{Z}_i$, for $i = 1, \ldots, n.$, and $\hat{\sigma}(\cdot)$ following steps 1 and 2 described above on the previous page in the two-step strategy.

(2) Estimate $F_Z^{-1}(\tau)$ by the sample τ-th quantile of $\hat{Z}_1, \ldots, \hat{Z}_n$, denoted by $\hat{F}_{\hat{Z}}^{-1}(\tau)$.

(3) Use $\hat{Z}_i$, $\hat{\sigma}(\cdot)$, and $\hat{F}_{\hat{Z}}^{-1}(\tau)$ in the proposed estimator $\hat{f}_Z(z)$, in (6), to estimate $f_Z(z)$. Thus, the estimation of $f_Z(F_Z^{-1}(\tau))$ is done through the proposed estimator $\hat{f}_{\hat{Z}}(\hat{F}_{\hat{Z}}^{-1}(\tau))$.

Finally, plug $\hat{f}_{\hat{Z}}(\hat{F}_{\hat{Z}}^{-1}(\tau))$ into $\Gamma(\hat{\beta}(\tau))$ in (22) to obtain $\hat{\Gamma}(\hat{\beta}(\tau))$. The performance of the proposed methodology to estimate the limiting variance of $\hat{\beta}(\tau)$ is examined, in sections 3.3 and 3.4, through both simulation data and an empirical analysis of the monthly inflation rate for the United States.

Given the properties of our estimator $\hat{f}_Z$, we conjecture that, although we use the estimate residuals, $\hat{Z}$, of the model (17), the resulting estimator $\hat{f}_{\hat{Z}}(z)$ in (6) is $\sqrt{n}$-consistent. However, it is difficult to prove this result and establish asymptotic normality. Therefore, we have decided to omit the proof for now, intending to report it in a future work. As a result, we shall not consider the consistency properties or any other large sample distribution for the estimator $\hat{f}_{\hat{Z}}(z)$. Instead, in Subsection 3.3, we report extensive simulation experiments that mimic various cases of practical interest to observe the performance of the proposed estimator compared with the traditional nonparametric kernel density estimator $\tilde{f}_{\hat{Z}}(z)$ in (2).

3 Numerical studies

In this Section, we will first investigate the performance of our density estimate compared with the traditional nonparametric kernel density estimate through a simulation study (see Subsection 3.1) and then apply our method to the S&P 500 Index data (refer to Subsection 3.2). In the second part, we will observe our proposed methodology's performance in estimating the limiting variance of the estimated coefficients in quantile regression with nonparametric ARCH errors through a simulation study (Subsection 3.3). Then, we will provide an empirical analysis of the monthly inflation rate of the United States as an illustration of this methodology (outlined in Subsection 3.4).

3.1 A simulation study for performance of $\hat{f}_Z(z)$

In this Subsection, we conduct a simulation study to examine the performance of the proposed estimator $\hat{f}_Z(z)$ in (6). Specifically, we compared our density estimate in (6) with the widely used nonparametric kernel density estimator $\tilde{f}_Z(z)$ in (2). We examine the effect of the sample size n and the use of different heteroscedastic models on the performance of the proposed density estimates. Our estimator uses the Gaussian kernel, which will also be used in all subsequent sections. In addition, the true density $f_Z(z)$ of Z is computed by numerically evaluating (4).

As mentioned in Subsection 2.1, since we are dealing with dependent data, we require the specific bandwidth $b_n = \tilde{\nu} b_n^*$ for some $\tilde{\nu} > 1$ in the traditional nonparametric kernel density estimator $\tilde{f}_Z(z)$. Based on our simulations, we observed that using only the rule-of-thumb bandwidth choice b_n^* tends to undersmooth the data due to the ignorance of the structure of dependence. Thus, for the purpose of improving the performance of $\tilde{f}_Z(z)$ in terms of mean-squared-error, we experimented with different bandwidths $b_n = \tilde{\nu} b_n^*$ for $\tilde{\nu} = 1.5, 1.8, 2$. We found that using $\tilde{\nu} = 2$ generates a better performance for this estimator under this context of models. Therefore, we use $b_n = 2b_n^*$ as the bandwidth for computing the density estimate $\tilde{f}_Z(z)$ in the simulation studies and applications from now. The rule-of-thumb bandwidth [Silverman (1986)] choice b_n^* is calculated using the `R` function `bw.nrd0`.

Consider observations Z_i, $1 \leq i \leq n$, from the following three models that impose different nonlinear structures on $\sigma(Z_{i-1})$, such as square-root heteroscedasticity (Model 1), absolute value heteroscedasticity (Model 2), and exponential-type heteroscedasticity (Model 3)[2]:

$$\text{Model 1:} \quad Z_i = \sigma\left(Z_{i-1}\right) e_i = \sqrt{\alpha_0 + \alpha_1 Z_{i-1}^2}\, e_i, \qquad \text{with } (\alpha_0, \alpha_1) = (0.1, 0.5); \tag{23}$$

$$\text{Model 2:} \quad Z_i = \sigma\left(Z_{i-1}\right) e_i = \Big(\alpha_0 + \alpha_1 \mid Z_{i-1} \mid\Big) e_i, \qquad \text{with } (\alpha_0, \alpha_1) = (0.1, 0.5); \tag{24}$$

$$\text{Model 3:} \quad Z_i = \sigma\left(Z_{i-1}\right) e_i = \exp\Big(\alpha_0 + \alpha_1 Z_{i-1}\Big) e_i, \qquad \text{with } (\alpha_0, \alpha_1) = (-1, 0.5). \tag{25}$$

For noise e_i, we assume that $e_i \overset{\text{iid}}{\sim} N(0,1)$. The normality assumption is quite common assumption in ARCH models. Here we have that Model 1 in (23) is the usual ARCH(1) model [Engle (1982)]. Model 2 in (24), the absolute value form, requires both parameters to be positive to assure a positive value for $\sigma(Z_{i-1})$ for all the observations i. Finally, Model 3 in (25), the exponential form, has the advantage that $\sigma(Z_{i-1})$ is positive for all values of α_0 and α_1; however, the data generated from such a model have infinite variance for any value of $\alpha_1 \neq 0$,

[2]We use these heteroscedasticity known specifications to assess the performance of the proposed estimator compared with the tradicional estimator. However in practice it is not posible to know the specific form for Z_i in advance.

see Engle (1982). Hence, to avoid the issue of infinite variance in this model 3, we use a negative number for α_0 in the simulation.

For the simulation, we use five sample sizes $n = 50, 100, 200, 400, 1000$, ranging from small to medium sample sizes. Also, we define a "warmup" period equal to 500 to reach an approximate stationary start for the simulated time series. For $\hat{f}_Z(z)$ in (6), we compute the mean-squared-error (MSE) as

$$\text{MSE}(\hat{f}_Z) = \text{average of 1000 realizations of } \left\{ \frac{q_Z(0.975) - q_Z(0.025)}{100} \sum_{j=1}^{100} [\hat{f}_Z(z_j) - f_Z(z_j)]^2 \right\}.$$

In the last expression we have that $q_Z(0.025)$ and $q_Z(0.975)$ are the 2.5% and 97.5% quantile of Z, respectively, and $z_1, \ldots, z_{100}$ are equally spaced points on the interval $[q_Z(0.025), q_Z(0.975)]$. The MSE for $\tilde{f}_Z(z)$ is defined similarly. In order to make an easy comparison, we also compute the relative MSE (RMSE) of $\hat{f}_Z$ as RMSE = $\text{MSE}(\hat{f}_Z)/\text{MSE}(\tilde{f}_Z)$, where a value of RMSE ≤ 1 indicates a better performance of our proposed estimator $\hat{f}_Z(z)$.

Table 1: MSE of $\hat{f}_Z$ and relative MSE (RMSE) of the proposed estimator of f_Z relative to the nonparametric kernel density estimator $\tilde{f}_Z$, at different sample sizes n. Numbers of RMSE ≤ 1 indicate better performance of $\hat{f}_Z(z)$.

	n	$\text{MSE}(\hat{f}_Z)$	RMSE
Model 1	50	0.0101	0.136
	100	0.0052	0.104
	200	0.0028	0.083
	400	0.0014	0.068
	1000	0.0006	0.061
Model 2	50	0.0318	0.152
	100	0.0165	0.117
	200	0.0092	0.098
	400	0.0051	0.092
	1000	0.0025	0.090
Model 3	50	0.0062	0.108
	100	0.0033	0.088
	200	0.0019	0.073
	400	0.0011	0.071
	1000	0.0006	0.080

Table 1 displays the MSE of $\hat{f}_Z$ in (6) and its RMSE to the nonparametric kernel density estimator $\tilde{f}_Z$ in (2). The following observations can be made from the table: Firstly, as the sample size increases, the MSE of $\hat{f}_Z$ decreases; this indicates an increasingly better performance of the proposed density estimates. Secondly, using $\hat{f}_Z(z)$ substantially reduces MSE with RMSE around 90% (i.e., RMSE $\leq$ 10%) for most cases. For instance, the MSE reduction for a sample size of $n = 100$ is approximately 89% for all the models considered. Thirdly, the larger the sample size, the smaller the RMSE, suggesting a good relative advantage of using $\hat{f}_Z(z)$ over the estimator $\tilde{f}_Z(z)$. This is because both estimators have bias terms that decrease as the sample size increases. While theoretically negligible, these bias terms may have some effects in finite samples. Since MSE = Bias^2 + Variance, as bias decreases, the variance plays a dominating role, leading to increasingly better relative performance of $\sqrt{n}$-consistent $\hat{f}_Z(z)$ than $\sqrt{nb_n}$-consistent $\tilde{f}_Z(z)$, where $b_n \propto n^{-1/5}$ decreases with sample size. Finally, the relative advantage of $\hat{f}_Z(z)$ over $\tilde{f}_Z(z)$ is consistent across the three models considered.

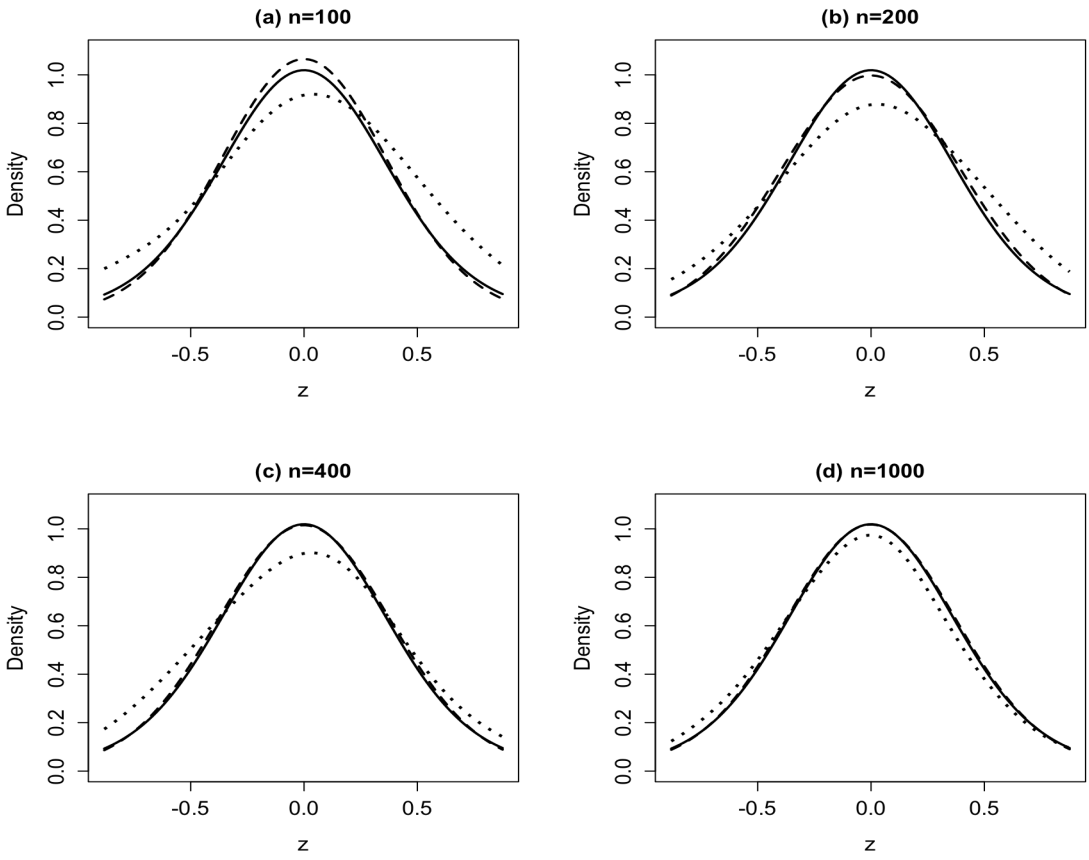


Figure 1: Four realizations (one realization on each figure) of density estimates with different sample sizes $n = 100, 200, 400, 1000$ from Model 1. In each plot, solid curve (——) is the true function, dotted curve ($\cdots$) is the usual nonparametric kernel estimates, and dashed curve (−−) is the proposed method.

To illustrate the performance of $\hat{f}_Z$, Figure 1 plots four realizations of density estimations from Model 1 with different sample sizes. In all the sample sizes considered, $\hat{f}_Z$ (dashed curve in all plots) matches very well the true densities (solid curve in all plots) for the realizations. On the other hand, there is an inferior performance for $\tilde{f}_Z$ (dotted curve in all plots) for a sample size of 100 and 200. When the sample size increases, $\tilde{f}_Z$ works reasonably well for density estimate. This behavior of poor performance of the usual nonparametric kernel density estimator, $\tilde{f}_Z$, occurs relatively often in our simulations.

3.2 Application of $\hat{f}_Z(z)$ to the S&P 500 index data

For this application, we use the proposed procedure to estimate the marginal density of the weekly log returns for Standard & Poor 500, hereafter S&P 500, stock market weekly index (available from `https://finance.yahoo.com/quote/%5ESPX/history?p=%5ESPX`), which has been of particular interest in financial econometrics. See Chan et al. (2013) or Zhao and Wu (2008), for example. The S&P 500 index monitors the stock performance of 500 of the largest companies listed on the United States stock exchanges. Thus, it is a widely followed equity index that allows investors to evaluate market performance by comparing the current stock price levels with past prices. Since we use weekly frequency data, we do not encounter the issue of missing data points, as is the case of the daily S&P 500 index. This fact over the daily index is due to stocks not being traded on weekends or holidays, only on so-called trading days. This results in missing data points in the daily time series, which typically require estimation before analysis.

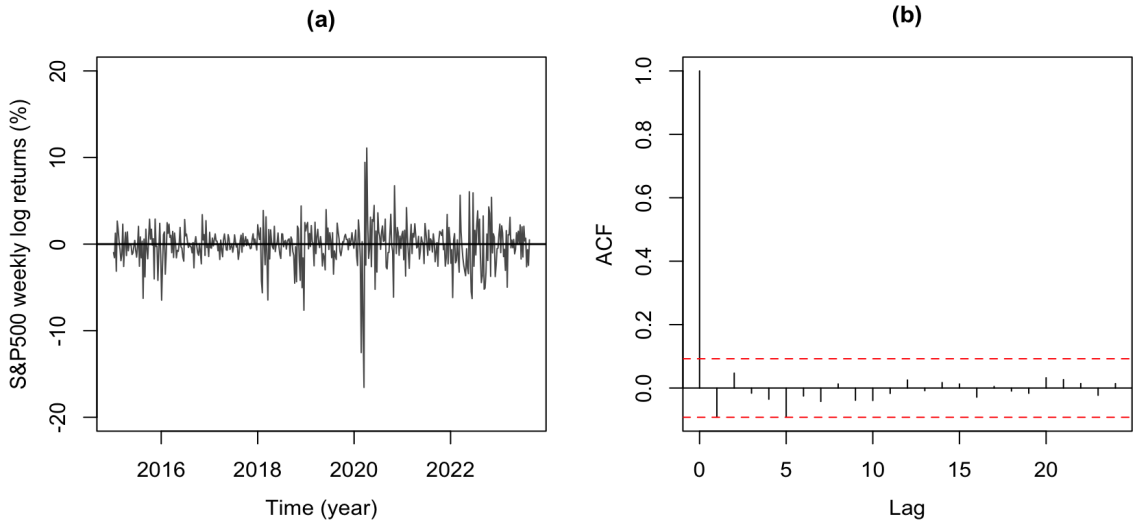


Figure 2: (a) The S&P 500 weekly log returns during 5 January 2015 - 21 August 2023, and (b) The estimated sample autocorrelation function (ACF) for the S&P 500 weekly log returns.

The data set consists of 451 weekly observations, $S_0, S_1, \ldots, S_{450}$, of S&P 500 index from 5 January 2015 to 21 August 2023. We work with the centered weekly log returns (as percentages) for S&P 500 index, that is, $Z_i = r_i - n^{-1}\sum_{i=1}^{n} r_i$, with $r_i = 100(\log(S_i) - \log(S_{i-1}))$ for $i = 1, \ldots, 450$. Ding et al. (1993) analyzed the daily log return series Z_i of the S&P 500 index, spanning from 3 January 1928 to 30 August 1991. Their findings

revealed that the S&P 500 returns Z_i are not independent and identically distributed. Moreover, they observed that Z_i exhibit some short-term memory and that there exists a small amount of predictability in stock returns. For additional insights on the serial dependence among S&P 500 returns, see, for example, Caporale and Gil-Alana (2004) and Awartani and Corradi (2005) and the references therein.

To analyze the behavior of the S&P 500 weekly log returns Z_i, we use Figure 2. From Figure 2(a), we can see that: (i) the time series is rather stable around zero. In fact, as the stationarity test of Kwiatkowski et al. (1992) (`R` command `kpss.test` in package `tseries`) has a p-value of 0.1, the null hypothesis of stationarity is not rejected at a level of significance 0.05; and (ii) the volatility of log returns is observed to change over time, where in particular, there are one period, specifically around 2020, where these volatilities are much higher than any other time frame considered here. Additionally, there is evidence of volatility clustering. Figure 2(b) plots the estimated sample autocorrelation function (ACF) for the S&P 500 weekly log returns, where the dashed red lines show the 95% confidence interval for these estimated sample ACF. We observe that this time series has no linear serial correlation, indicating the absence of conditional-mean dynamics for the process. However, upon analyzing the ACF of the squared log returns (which is not shown here), we notice some significant autocorrelations. Thus, there is a higher-order serial dependence structure in the log returns, verifying the existence of ARCH effects in the time series.

The analysis from Figure 2 previously given suggests that a suitable model for this dataset should possess a volatility structure that varies with time, as indicated by the nonparametric model in (1). Hence, the proposed estimator $\hat{f}_Z(z)$ in (6) can be applied to estimate the marginal density for the S&P 500 weekly log returns.

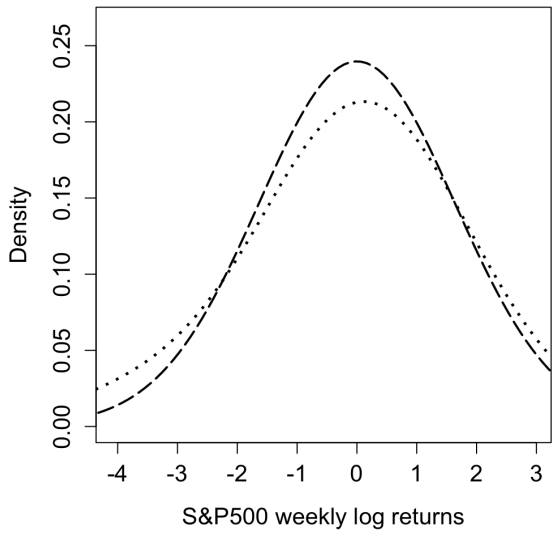


Figure 3: The estimated density curves of the S&P 500 weekly log returns. The dotted curve ($\cdots$) is the usual nonparametric kernel estimate, and the dashed curve ($--$) is the proposed method.

Figure 3 presents the estimated density curves for the S&P 500 weekly log returns using both the usual nonparametric kernel density estimate $\tilde{f}_Z(z)$ in (2) (dotted curve), and our density estimate $\hat{f}_Z(z)$ in (6) (dashed curve) for comparison. The density curves are evaluated at 100 equally spaced points between the 2.5% and 97.5%

quantiles of the log returns in the data. The two estimated density curves seem to have a similar shape for log returns. However, $\tilde{f}_Z(z)$ tends to show higher estimated density values for low and high log-returns, and lower values for the intermediate range of log-returns, compared to our density estimate $\hat{f}_Z(z)$. This behavior in the density estimates is similar to that observed in Figure 1 for the specific density estimates in the simulation study in Subsection 3.1.

3.3 A simulation study for performance of $\hat{f}_Z(z)$ in the context of Quantile regression

In this Subsection, we conduct an extensive simulation study designed to investigate the performance of our proposed density estimate $\hat{f}_Z$ in the context of quantile regression with heteroskedastic form for the error term. Specifically, we compared our density estimate in (6) with the widely used nonparametric kernel density estimator $\tilde{f}_Z(z)$ in (2) to estimate the limiting variance of the estimated coefficients in a quantile model with different heteroskedastic forms for the error term noticed by Z.

Each model under consideration is of the form

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + Z_i, \quad i = 1, \ldots, n. \tag{26}$$

For the error term Z_i, similar to Subsection 3.1, we consider three widely used forms of heteroscedasticity to set the three models to study.

$$\text{Heteroscedasticity 1:} \quad Z_i = \sigma(Z_{i-1})\, e_i = \sqrt{\alpha_0 + \alpha_1 Z_{i-1}^2}\, e_i, \quad \text{with } (\alpha_0, \alpha_1) = (0.1, 0.5); \tag{27}$$

$$\text{Heteroscedasticity 2:} \quad Z_i = \sigma(Z_{i-1})\, e_i = \Big(\alpha_0 + \alpha_1 \mid Z_{i-1} \mid\Big) e_i, \quad \text{with } (\alpha_0, \alpha_1) = (0.1, 0.5); \tag{28}$$

$$\text{Heteroscedasticity 3:} \quad Z_i = \sigma(Z_{i-1})\, e_i = \exp\Big(\alpha_0 + \alpha_1 Z_{i-1}\Big) e_i, \quad \text{with } (\alpha_0, \alpha_1) = (-1, 0.5). \tag{29}$$

The noise e_i is taken to be $e_i \overset{\text{iid}}{\sim} N(0,1)$, the true parameter is set to be $\beta = (\beta_0, \beta_1, \beta_2)^T = (10, -4, 2)^T$, and the covariates X_1 and X_2, in vector $X_i = (1, X_{1i}, X_{2i})$, are generated by [see Zhao (2001)]:

$$X_1 = U + 0.2V, \quad X_2 = 0.2U + V, \quad U \sim N(5, 9), \quad V \sim U(0, 4).$$

Thus, we have the following three distinct models.

Model 1: This model is formed by the linear regression model in (26) and the square-root heteroscedasticity 1 in (27).

Model 2: This model resembles Model 1, but with the absolute value heteroscedasticity 2 in (28).

Model 3: This model also employs the linear regression model in (26) but involves the exponential-type heteroscedasticity 3 in (29). As we mentioned in Subsection 3.1, this heteroscedasticity 3 is positive for all values of α_0 and α_1; however, the data generated from such a model have infinite variance for any value of $\alpha_1 \neq 0$, see Engle (1982). Hence, to avoid the issue of infinite variance in this heteroscedasticity 3, we use a negative number for α_0 in the simulation.

The estimation of each of the three models, for fixed value $\tau \in (0,1)$, is done by using the two-step strategy described in Subsection 2.2. This is, for each model, in step 1, we compute the estimation of $\beta(\tau) = (\beta_0 + F_Z^{-1}(\tau), \beta_1, \beta_2)^T$ using the quantile regression estimator in (19) (`R` command `qr` in package `quantreg`). Now, in step 2, we calculate the regression residuals using $\hat{Z}_i = Y_i - X_i^T\hat{\beta}(\tau)$, for $i = 1, \ldots, n$. To ensure a fair comparison among the density estimators, it is necessary to center $\hat{Z}_i$ around its mean. This is because these estimated residuals do not have zero mean, but by construction, the stationary density of the true residuals has. Therefore, we use this approach for this simulation and the application in the following Subsection. Henceforth, the centered estimated residuals will be referred to as $\hat{Z}_i$.

Now, based on $\hat{Z}_i$: (i) we use the kernel smoothing estimate given in (7) to estimate the heteroskedastic form $\sigma(\cdot)$ and (ii) the proposed estimator $\hat{f}_{\hat{Z}}$ to estimate f_Z. The quantity $\hat{Z}_i$ is specific to the τ-th quantile. For each of the three models considered, we are interested in estimating the limiting variance $\hat{\Gamma}(\hat{\beta}_j(\tau))$, for $j = 0, 1, 2$, in (22). To do this, based on $\hat{Z}_1, \ldots, \hat{Z}_n$, we compare the following two methods:

(1) *OM*: (Our method) We utilize the proposed estimator $\hat{f}_{\hat{Z}}(z)$ evaluated at the quantile of interest $\hat{z}_\tau = \hat{F}_{\hat{Z}}^{-1}(\tau)$, to estimate the quantity $1/f_Z(F_Z^{-1}(\tau))$ present in the expression of $\Gamma(\hat{\beta}(\tau))$ in (22). Then, we plug $1/\hat{f}_{\hat{Z}}(\hat{z}_\tau)$ into $\Gamma(\hat{\beta}(\tau))$ to obtain $\hat{\Gamma}(\hat{\beta}(\tau))$.

(2) *NKDE*: Simalar as OM, but in this case, we use the nonparametric kernel density estimator $\tilde{f}_{\hat{Z}}(z)$, in (2), evaluated at the quantile of interest $\hat{z}_\tau$, to estimate $1/f_Z(F_Z^{-1}(\tau))$. Then, we plug $1/\tilde{f}_{\hat{Z}}(\hat{z}_\tau)$ into $\Gamma(\hat{\beta}(\tau))$ to obtain $\hat{\Gamma}_{\text{NKDE}}(\hat{\beta}(\tau))$.

We performed a simulation study to compare the two methods mentioned above in order to estimate the limiting variance of $\hat{\beta}_j(\tau)$ under the three models considered; the number of replications in each method is $m = 1000$. For each run, we use sample sizes of $n = 100, 400$, and 1000. We consider five quantile levels: 0.1, 0.25, 0.5, 0.75, and 0.9. The results for these sample sizes are the only ones we are presenting, as we acquired similar results for sample sizes of 50 and 200. The mean-squared-error (MSE) is a commonly used criterion to evaluate the performance of estimators. Thus, for any $0 < \tau < 1$ and using the proposed density estimator $\hat{f}_Z(z)$ in (6) based on the estimated residuals of the model $\hat{Z}_1, \ldots, \hat{Z}_n$ (OM method), we compute the MSE of the estimated asymptotic variance of $\hat{\beta}_j(\tau)$, for $j = 0, 1, 2$, as

$$\text{MSE}\left(\hat{\Gamma}(\hat{\beta}_j(\tau))\right) = \frac{\sum_{i=1}^{m}\left[\hat{\Gamma}^{(i)}(\hat{\beta}_j(\tau)) - \Gamma(\hat{\beta}_j(\tau))\right]^2}{m},$$

where $\hat{\Gamma}^{(i)}(\hat{\beta}_j(\tau))$ is the estimated asymptotic variance of $\hat{\beta}_j(\tau)$ on the i-th sample and $\Gamma(\hat{\beta}_j(\tau))$ is the true asymptotic variance of $\hat{\beta}_j(\tau)$, respectively. The $\Gamma(\hat{\beta}_j(\tau))$ value is the diagonal value obtained from the matrix $\Gamma(\cdot)$ in (22). The MSE for the estimation of the asymptotic variance of $\hat{\beta}_j(\tau)$ by utilizing the nonparametric kernel density estimator (NKDE method) in (2) is defined similarly, this MSE is represented by $\text{MSE}\left(\hat{\Gamma}_{\text{NKDE}}(\hat{\beta}_j(\tau))\right)$. In order to make the results more readable, we also compute the relative MSE (RMSE), defined as

$$\text{RMSE}\left(\hat{\Gamma}(\hat{\beta}_j(\tau))\right) = \frac{\text{MSE}\left(\hat{\Gamma}(\hat{\beta}_j(\tau))\right)}{\text{MSE}\left(\hat{\Gamma}_{\text{NKDE}}(\hat{\beta}_j(\tau))\right)},$$

with RMSE ≤ 1 indicating a better performance in estimating the asymptotic variance of $\hat{\beta}_j(\tau)$, for $j = 0, 1, 2$, using our proposed method OM. The simulation results are summarized in Table 2.

Table 2 shows the MSE ($\times 100$) of the estimated asymptotic variance $\hat{\Gamma}(\hat{\beta}_j(\tau))$ by utilizing the proposed method (OM) and the RMSEs. We make the following observations:

(1^0) In quantiles close to $\tau = 0.5$ (specifically for τ=0.25, 0.50, and 0.75), the use of our method (OM) results in a substantial reduction in MSE with RMSE below 91% for most cases considered. For example, for $\hat{\beta}_1$ in Model 1 with $\tau = 0.5$ and $n = 100$, the reduction in the MSE is about 84%. For values of τ of 0.1 and 0.9, when we have a small sample size $n = 100$, the NKDE method performs better than our method (OM). However, as the sample size increases to 400, the OM method shows a reduction in MSE with an RMSE of around 36% for most cases. This reduction in MSE for our method is even better for $n = 1000$.

(2^0) The performance of our method (OM) to estimate $\hat{\Gamma}(\hat{\beta}_j(\tau))$, for $j = 0, 1, 2$, is consistent across the three models and sample sizes considered, and it is even better in the cases of values of τ close to 0.5. This may be because more observations are available as τ gets closer to the center.

(3^0) The larger the sample size, the smaller the RMSE, and thus there is a relative advantage of using $\hat{f}_{\hat{Z}}(z)$ over $\tilde{f}_{\hat{Z}}(z)$ to estimate the asymptotic variance of $\hat{\beta}_j(\tau)$'s.

In conclusion, substantial efficiency gain can be achieved by using our proposed density estimator $\hat{f}_{\hat{Z}}(z)$ in (6) (OM method), based on the estimated residuals of the model, to estimate the asymptotic variance of $\hat{\beta}_j(\tau)$'s in the quantile regression model compared with the NKDE method.

In Subsection 2.2, it was mentioned that the estimation of $\Gamma(\hat{\beta}(\tau))$ in (22) depends on the reciprocal of the density function of the model error terms Z, evaluated at the quantile of interest $z_\tau = F_Z^{-1}(\tau)$, which would be $1/f_Z^2(z_\tau)$. This means that the accuracy of this estimation $\hat{\Gamma}(\hat{\beta}(\tau))$ varies based on the accuracy of the point estimation $f_Z(z_\tau)$. So, even though, in some cases, the estimation of the limiting variance appears to be good, a bias in the estimation of $f_Z(z_\tau)$ can lead to incorrect conclusions about the quality of the estimation of $\Gamma(\hat{\beta}(\tau))$. To investigate the behavior of the point estimation of $f_Z(z_\tau)$, we do the following: First, we compute a ratio using the formula Ratio$= \hat{f}_{\hat{Z}}(\hat{z}_\tau) / \tilde{f}_{\hat{Z}}(\hat{z}_\tau)$, with $\hat{z}_\tau = \hat{F}_{\hat{Z}}^{-1}(\tau)$. This is done in Table 3. Second, based on the expression (16), we calculate an approximation of bias for the nonparametric kernel density estimator $\tilde{f}_{\hat{Z}}$ in Table 4.

Table 3 displays the ratio, described in the last paragraph, at different sample sizes and levels of τ for the three models considered. A value of Ratio> 1 indicates that $\tilde{f}_{\hat{Z}}(\hat{z}_\tau) < \hat{f}_{\hat{Z}}(\hat{z}_\tau)$. The reported values of the ratio are based on averaging over the $m = 1000$ realizations of the simulation study described above. It can be seen that for a small sample size ($n = 100$) and quantiles close to $\tau = 0.5$, as Ratio> 1, the point estimation using the nonparametric kernel density $\tilde{f}_{\hat{Z}}(\hat{z}_\tau)$ tends to be less than estimation $\hat{f}_{\hat{Z}}(\hat{z}_\tau)$. Also, when $n = 100$ and we consider quantiles $\tau = 0.1$ and $\tau = 0.9$, we have Ratio< 1, indicating that $\tilde{f}_{\hat{Z}}(\hat{z}_\tau) > \hat{f}_{\hat{Z}}(\hat{z}_\tau)$. This behavior in the ratio values might be due to the bias present in the estimator $\tilde{f}_{\hat{Z}}$, which diminishes when the sample size increases, as we see for the sample size of 1000 where the ratio is approximately equal to 1 in all cases. Thus, in Table 4, we compute an approximation of this bias of $\tilde{f}_{\hat{Z}}$.

Table 2: MSE (×100) of the estimation for the asymptotic variance of $\hat{\beta}_j(\tau)$, for $j = 0, 1, 2$, by utilizing the proposed estimator $\hat{f}_{\hat{Z}}(z)$ (OM method) and relative MSE (RMSE). RMSE ≤ 1 indicates a better performance in estimating the asymptotic variance of $\hat{\beta}_j(\tau)$, using $\hat{f}_{\hat{Z}}(z)$.

			MSE (×100)			RMSE		
	n	τ	$\hat{\Gamma}(\hat{\beta}_0(\tau))$	$\hat{\Gamma}(\hat{\beta}_1(\tau))$	$\hat{\Gamma}(\hat{\beta}_2(\tau))$	$\hat{\Gamma}(\hat{\beta}_0(\tau))$	$\hat{\Gamma}(\hat{\beta}_1(\tau))$	$\hat{\Gamma}(\hat{\beta}_2(\tau))$
Model 1	100	0.10	303.0325	0.1626	4.3804	1.171	1.245	1.185
		0.25	35.5103	0.0209	0.4688	0.361	0.372	0.353
		0.50	22.6300	0.0126	0.2791	0.152	0.160	0.141
		0.75	45.2515	0.0241	0.5827	0.391	0.396	0.375
		0.90	298.5324	0.1593	4.2161	1.123	1.146	1.103
	400	0.10	93.5466	0.0509	1.3160	0.771	0.789	0.742
		0.25	9.1675	0.0048	0.1237	0.345	0.353	0.340
		0.50	4.1599	0.0023	0.0546	0.113	0.117	0.104
		0.75	8.8603	0.0046	0.1140	0.368	0.362	0.348
		0.90	81.6479	0.0410	1.1311	0.633	0.633	0.606
	1000	0.10	34.9604	0.0181	0.4972	0.506	0.505	0.490
		0.25	3.3439	0.0018	0.0463	0.332	0.342	0.334
		0.50	1.8298	0.0010	0.0222	0.098	0.104	0.086
		0.75	3.5278	0.0019	0.0459	0.356	0.363	0.343
		0.90	39.3961	0.0205	0.5650	0.494	0.492	0.474
Model 2	100	0.10	9.8656	0.0053	0.1352	1.162	1.197	1.128
		0.25	1.1899	0.0007	0.0152	0.386	0.388	0.374
		0.50	0.6416	0.0004	0.0080	0.172	0.179	0.159
		0.75	1.0435	0.0006	0.0140	0.393	0.392	0.376
		0.90	8.9040	0.0051	0.1305	1.085	1.159	1.070
	400	0.10	2.8060	0.0014	0.0396	0.679	0.667	0.656
		0.25	0.2510	0.0001	0.0033	0.392	0.386	0.384
		0.50	0.1362	0.0001	0.0017	0.132	0.134	0.118
		0.75	0.2481	0.0001	0.0031	0.413	0.430	0.406
		0.90	2.8315	0.0015	0.0405	0.632	0.628	0.611
	1000	0.10	1.3435	0.0007	0.0194	0.552	0.552	0.540
		0.25	0.0919	0.0001	0.0012	0.376	0.396	0.381
		0.50	0.0550	0.0001	0.0007	0.116	0.119	0.110
		0.75	0.0969	0.0001	0.0013	0.412	0.395	0.394
		0.90	1.4433	0.0007	0.0210	0.549	0.549	0.540
Model 3	100	0.10	124.0774	0.0638	1.6628	1.153	1.189	1.124
		0.25	17.5952	0.0097	0.1992	0.343	0.352	0.303
		0.50	10.3859	0.0061	0.1347	0.132	0.146	0.125
		0.75	21.4572	0.0107	0.2261	0.378	0.364	0.324
		0.90	136.5286	0.0738	1.8223	1.157	1.238	1.147
	400	0.10	30.0570	0.0154	0.4176	0.618	0.621	0.595
		0.25	4.0737	0.0023	0.0486	0.288	0.315	0.270
		0.50	3.0049	0.0014	0.0336	0.122	0.119	0.102
		0.75	3.9036	0.0021	0.0471	0.291	0.312	0.272
		0.90	33.2228	0.0182	0.4591	0.582	0.589	0.548
	1000	0.10	13.9714	0.0072	0.1966	0.529	0.525	0.507
		0.25	1.6179	0.0009	0.0195	0.257	0.261	0.228
		0.50	1.3688	0.0008	0.0166	0.120	0.127	0.104
		0.75	1.4945	0.0008	0.0183	0.248	0.267	0.228
		0.90	17.2736	0.0088	0.2454	0.505	0.506	0.490

Table 3: Ratio= $\hat{f}_{\hat{Z}}(\hat{z}_\tau)/\tilde{f}_{\hat{Z}}(\hat{z}_\tau)$, with $\hat{z}_\tau = \hat{F}_{\hat{Z}}^{-1}(\tau)$, at different sample sizes and different levels of τ for the three models considered. Ratio>1 indicates that $\tilde{f}_{\hat{Z}}(\hat{z}_\tau) < \hat{f}_{\hat{Z}}(\hat{z}_\tau)$.

		Ratio		
n	τ	Model 1	Model 2	Model 3
100	0.10	0.896	0.903	0.911
	0.25	1.124	1.124	1.109
	0.50	1.218	1.203	1.188
	0.75	1.125	1.121	1.103
	0.90	0.894	0.902	0.904
400	0.10	0.921	0.927	0.940
	0.25	1.082	1.076	1.069
	0.50	1.146	1.135	1.118
	0.75	1.082	1.080	1.067
	0.90	0.919	0.926	0.940
1000	0.10	0.940	0.948	0.961
	0.25	1.058	1.056	1.048
	0.50	1.105	1.094	1.080
	0.75	1.059	1.059	1.048
	0.90	0.939	0.950	0.961

From the expression of bias for the nonparametric kernel density estimator $\tilde{f}_Z(z)$ in Corollary 1, we have that $\mathrm{C}_Z(z)$ in (16) represents a constant coefficient term, with respect to n, for the bias of the estimator $\tilde{f}_Z(z)$. The term $\mathrm{C}_Z(z)$ ignores the calculation of bandwidth b_n, which determines the bias for specific sample sizes. We do this to have a clear idea of the direction of the bias of $\tilde{f}_Z(z)$ and to observe for what values of τ this bias shows a bigger magnitude. Thus, for the three models considered above, we get a theoretical approximation for the bias of $\tilde{f}_Z$ through the approximation of $\mathrm{C}_Z(z)$ in (16). From now, we refer to $\mathrm{C}_Z(z)$ as bias approximation of $\tilde{f}_Z(z)$.

Table 4 presents the bias approximation $\mathrm{C}_Z(z)$ evaluated at specific quantiles for the three models considered, which possess different forms of heteroscedasticity. To compute this approximation, we numerically evaluate (16). Each value z_τ in Table 4 corresponds to the τ-th quantile of error terms Z in each case. It is possible to observe that across all three models, the estimator $\tilde{f}_Z$ shows the following facts: Firstly, it tends to overestimate (positive bias) the true density f_Z for quantiles around $\tau \approx 0.1$ and $\tau \approx 0.9$. Here, it is useful to note that given the heteroscedasticity structure imposed on Z, large negative values of Z are associated with $\tau \approx 0.1$, and large positive values of Z are associated with $\tau \approx 0.9$. Secondly, the estimator $\tilde{f}_Z$ underestimates (negative bias) f_Z for quantiles close to 0.5 where $z_\tau = F_Z^{-1}(\tau) = 0$. Finally, as the quantile level τ approaches 0.5, the bias increases in magnitude.

The bias approximation pattern of $\tilde{f}_Z$ in Table 4 can explain the behavior of the ratio values in Table 3. This is, in Table 3, when τ is around values close to 0.5, we will tend to have a negative bias of $\tilde{f}_Z$, causing in some cases that $\tilde{f}_Z(\hat{z}_\tau) < \hat{f}_Z(\hat{z}_\tau)$. This results in a Ratio= $\hat{f}_{\hat{Z}}(\hat{z}_\tau)/\tilde{f}_{\hat{Z}}(\hat{z}_\tau) > 1$. On the other hand, when τ is around values close to 0.1 or 0.9, $\tilde{f}_Z$ will show a positive bias causing in some cases that $\tilde{f}_Z(\hat{z}_\tau) > \hat{f}_Z(\hat{z}_\tau)$. This generates a Ratio$<1$.

Based on Tables 3 and 4, we can observe two specific cases for small sample sizes ($n = 100$). Firstly, if τ is close to 0.5 (i.e. $z \approx 0$), then the bias of $\tilde{f}_Z(z)$ is negative and $\tilde{f}_Z(\hat{z}_\tau) < \hat{f}_Z(\hat{z}_\tau)$ for most of the replications in the simulation study. As a result, for $j = 0, 1, 2$, $\hat{\Gamma}_{\text{NKDE}}(\hat{\beta}_j(\tau)) > \hat{\Gamma}(\hat{\beta}_j(\tau))$ in most cases. Thus in this scenario, using

Table 4: Approximation of bias of the widely used nonparametric kernel density estimator $\tilde{f}_Z(z)$ in (6), at different quantiles τ, $z_\tau = F_Z^{-1}(\tau)$, for the three models considered.

	τ	z_τ	$\mathrm{C}_Z(z_\tau)$
Model 1	0.10	-0.524	1.392
	0.25	-0.266	-1.228
	0.50	0.000	-3.787
	0.75	0.267	-1.204
	0.90	0.526	1.400
Model 2	0.10	-0.208	25.759
	0.25	-0.104	-16.924
	0.50	0.000	-70.500
	0.75	0.104	-16.484
	0.90	0.209	25.820
Model 3	0.10	-0.475	1.541
	0.25	-0.245	-1.522
	0.50	0.000	-4.728
	0.75	0.245	-1.513
	0.90	0.480	1.563

$\tilde{f}_Z$ tends to overestimate the limiting variance $\Gamma(\hat{\beta}_j(\tau))$. Secondly, if $\tau \approx 0.1$ or $\tau \approx 0.9$, then the bias of $\tilde{f}_Z(z)$ is positive and $\tilde{f}_Z(\hat{z}_\tau) > \hat{f}_Z(\hat{z}_\tau)$ in most of the replications in the simulation study. In this case, we will observe that $\hat{\Gamma}_{\text{NKDE}}(\hat{\beta}_j(\tau)) < \hat{\Gamma}(\hat{\beta}_j(\tau))$, which indicates a tendency to underestimate the limiting variance $\Gamma(\hat{\beta}_j(\tau))$ when using $\tilde{f}_Z$. These two specific cases can try to explain the pattern we observe in the performance of the confidence intervals for $\hat{\beta}(\tau)$ in Tables 5 and 6 below. An example of this behavior is illustrated in Figure 4.

To illustrate the performance of $\hat{f}_{\hat{Z}}$ for estimating the true density of model residuals f_Z, Figure 4 plots six realizations of density estimations of f_Z based on the estimated residuals from Models 1 and 2 with $n = 100$, and $\tau = 0.1, 0.5$ and 0.9. First, we observe that the proposed method (dashed curve in all plots) closely matches the true curves of the residuals (solid curve in all plots) somewhat well for the six realizations. By contrast, the nonparametric kernel density estimate $\tilde{f}_{\hat{Z}}$ (dotted curve in all plots) performs poorly in most realizations. However, this estimator $\tilde{f}_{\hat{Z}}$ works reasonably well when the sample size is increased (not shown in the plot) but not as well as our method. Second, we get a more accurate estimate for the quantity $f_Z(z_\tau)$ (the circle symbol in the solid curve in all plots) by using our proposed method (the square symbol in the dashed curve in all plots) compared with the estimation of this quantity using $\tilde{f}_{\hat{Z}}$ (the triangle symbol in the dotted curve in all plots) for the six realizations. It is noteworthy that the poor performance in the estimation of $f_Z(z_\tau)$ when we use $\tilde{f}_{\hat{Z}}$ can be due to the bias present in this estimator. This is, for small sample sizes, $\tilde{f}_{\hat{Z}}$ tends to overestimate $f_Z(z_\tau)$ at quantiles close to $\tau = 0.1$ or 0.9, and underestimate it at values of τ close to 0.5 as described above in the last paragraph.

To conclude this Subsection, for the simulation study, we present a confidence interval performance for $\hat{\beta}_0(\tau)$, $\hat{\beta}_1(\tau)$, and $\hat{\beta}_2(\tau)$ when we estimate the corresponding limiting variance using the two methods described earlier: (i) Our proposed method (OM) using the $\hat{f}_Z$, in (6), and (ii) the method NKDE using the estimator $\tilde{f}_Z$, in (2). Table 5 and Table 6 report results of coverage probability and lengths of confidence intervals, respectively, under the two methods to estimate the limiting variance $\hat{\Gamma}(\hat{\beta}_j(\tau))$, for $j = 0, 1, 2$. We exclusively focus on five quantile levels (τ) under the models mentioned above for $n = 100, 400$ and 1000. In all cases, we present results for confidence intervals that are intended to have a coverage of 0.95. The reported estimated coverage probability (see Table 5) and lengths (see Table 6) for the estimated confidence interval are based on averaging over the 1000 realizations

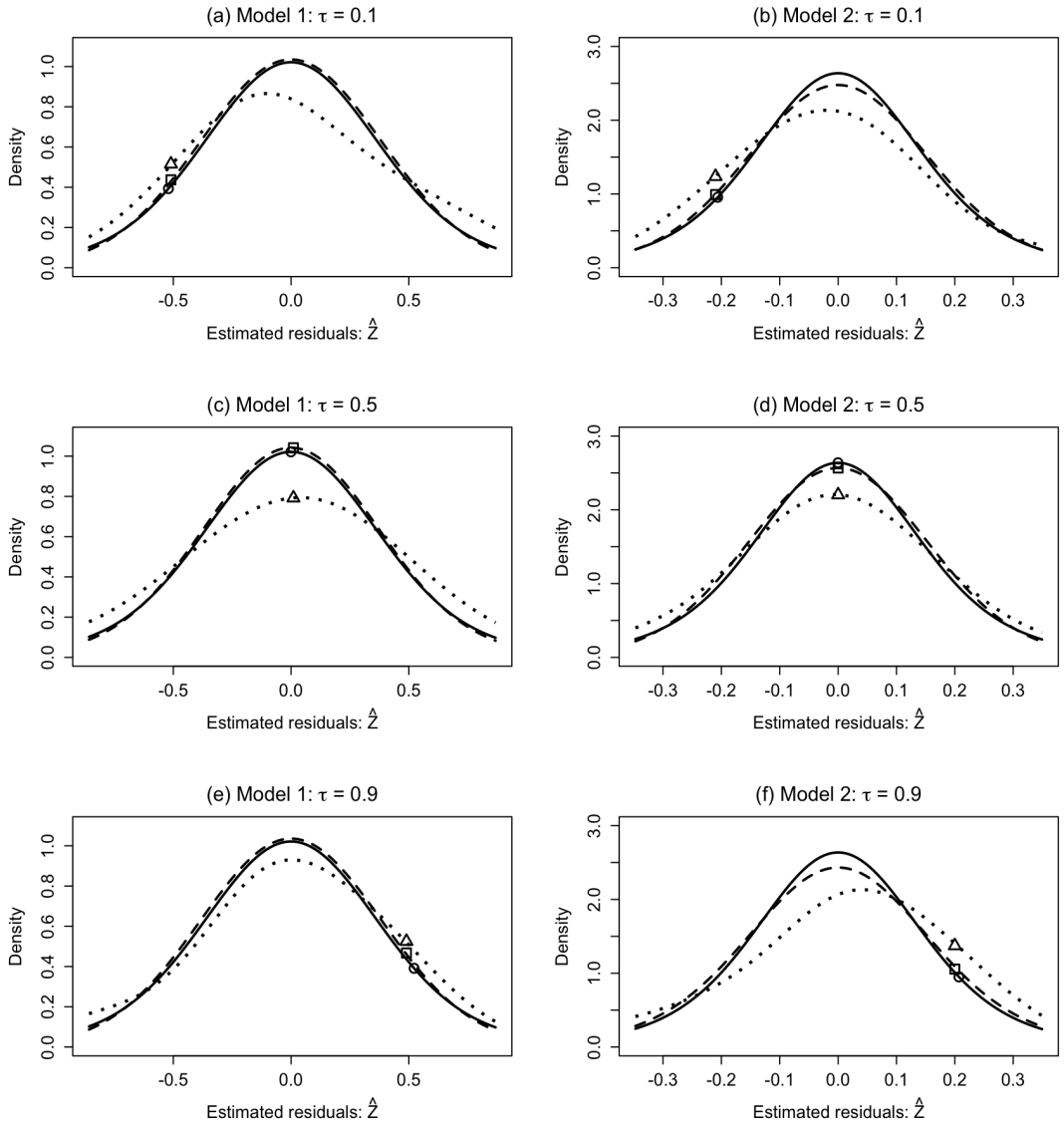


Figure 4: Six realizations (one realization on each figure) of density estimates for the estimated residuals from Models 1 and 2 with $\tau = 0.1, 0.5, 0.9$ and sample size $n = 100$. In each plot, solid curve (——) is the true function with the circle symbol ($\circ$) representing the true value of quantity $f_Z(z_\tau)$, where $z_\tau = F_Z^{-1}(\tau)$, dotted curve ($\cdots$) is the usual nonparametric kernel estimates with the triangle symbol ($\triangle$) representing the estimation of $f_Z(z_\tau)$ using this method (NKDE), and dashed curve ($--$) is the proposed method with the square symbol ($\square$) representing the estimation of $f_Z(z_\tau)$ using our method (OM)

of this simulation study.

The results presented in Table 5 indicate that the average 95% confidence interval coverage rate is a bit low for the smallest sample size ($n = 100$) in both methods but gets closer to the nominal 0.95 level as n increases for all quantiles, models, and estimated parameters. For quantiles $\tau = 0.25, 0.5$ and 0.75, both methods perform quite well as the empirical coverage probability is close to 95%. However, in most cases, our method (OM) performs

little better than the NPKDE method. For quantiles $\tau = 0.1$ and 0.9, our approach (OM) outperforms the NPKDE method in all $\hat{\beta}(\tau)$'s. For example, under Model 1, for the confidence interval of $\hat{\beta}_1(\tau)$, with $n = 100$ and $\tau = 0.1$, our method (OM) exhibits an empirical coverage of 93%, whereas the NPKDE method only has a coverage rate of 90%. The relative advantage of the proposed method (OM) over the NPKDE approach is consistent across the three models, which consider different forms of heteroscedasticity.

From Table 6, we make the following observations:

(1^0) It can be seen that the mean lengths of confidence intervals are similar for both approaches when considering $\tau = 0.25, 0.5$, and 0.75. However, the proposed method (OM) generally exhibits lower standard errors for these average lengths than the NPKDE method. For instance, in Model 1 with $n = 100$ and $\tau = 0.75$, the mean lengths of the confidence interval for $\hat{\beta}_2(\tau)$ are similar for both methods, at 0.20 and 0.22, respectively. However, our method (OM) has a lower standard error (the bracketed number) of 0.028 compared to 0.034 for the NPKE method. Notably, in this example, both methods have a similar empirical coverage probability around 95% (See Table 5).

(2^0) For quantiles $\tau = 0.1$ and 0.9, the proposed approach (OM) exhibits wider confidence intervals than the NPKE approach but with similar standard errors for the mean length. This behavior can be explained by the pattern observed in Tables 3 and 4 for the estimation of the limiting variance $\Gamma(\hat{\beta}_j(\tau))$ when using $\tilde{f}_Z$. This is, for these values of τ, we have $\hat{\Gamma}_{\text{NKDE}}(\hat{\beta}_j(\tau)) < \hat{\Gamma}(\hat{\beta}_j(\tau))$, resulting in wider confidence intervals when using the proposed estimator in the simulation study.

(3^0) As n increases, both the average lengths and their respective standard errors decrease.

Overall, based on the results from Table 5 and Table 6, we can conclude that it is better to use our method (OM) compared to the NKDE method, to estimate the limiting variance of $\hat{\beta}_j(\tau)$'s , as it leads to better performance of the confidence interval for the estimated coefficients in the quantile regression model.

3.4 Quantile regression application to the monthly inflation rate for the United States

The behavior of prices, in some way, the change of prices (inflation rate), is a key variable in the economy, which determines the population's purchasing power and affects the investment decisions of different agents. Due to its relevance, institutions such as central banks, with scheme inflation target, try to keep the inflation rate low and stable in order to preserve the purchasing power of money and thus contribute to the economy producing at the maximum level of its capacities. Therefore, this is one of the most relevant economic series, especially for countries with target inflation, which follows the changes (variations) in the price of a basket of goods and services representative of the consumption of households (See IMF (2004)). The percentage change of the consumer price index (CPI) is a typically used measure of the inflation rate, which is a statistical measure, over time, of the prices of goods and services in major expenditure groups. Thus, in this Subsection, to apply the proposed density estimate in the context of quantile regression with nonparametric ARCH errors, we analyze the monthly inflation time series of the total CPI of the United States.

Table 5: Comparison of empirical coverage rates of the 95% confidence intervals for the estimated coefficients $\hat{\beta}_0(\tau)$, $\hat{\beta}_1(\tau)$ and $\hat{\beta}_2(\tau)$ under the three models considered at different sample sizes n and quantile levels τ.

			$\hat{\beta}_0(\tau)$		$\hat{\beta}_1(\tau)$		$\hat{\beta}_2(\tau)$	
	n	τ	OM	NKDE	OM	NKDE	OM	NKDE
Model 1	100	0.10	0.91	0.88	0.93	0.90	0.93	0.90
		0.25	0.94	0.96	0.95	0.97	0.94	0.96
		0.50	0.95	0.98	0.94	0.97	0.96	0.98
		0.75	0.94	0.96	0.95	0.96	0.94	0.96
		0.90	0.90	0.87	0.94	0.91	0.93	0.90
	400	0.10	0.93	0.91	0.93	0.91	0.94	0.92
		0.25	0.94	0.96	0.94	0.96	0.94	0.96
		0.50	0.95	0.97	0.96	0.98	0.95	0.97
		0.75	0.94	0.96	0.95	0.96	0.94	0.95
		0.90	0.94	0.93	0.95	0.93	0.94	0.92
	1000	0.10	0.94	0.93	0.94	0.92	0.94	0.93
		0.25	0.95	0.96	0.95	0.96	0.95	0.97
		0.50	0.94	0.96	0.95	0.97	0.96	0.97
		0.75	0.96	0.97	0.95	0.96	0.95	0.96
		0.90	0.93	0.91	0.94	0.93	0.94	0.93
Model 2	100	0.10	0.90	0.86	0.92	0.89	0.90	0.87
		0.25	0.94	0.96	0.95	0.96	0.95	0.96
		0.50	0.94	0.98	0.96	0.98	0.94	0.97
		0.75	0.94	0.97	0.94	0.96	0.95	0.97
		0.90	0.91	0.87	0.91	0.87	0.92	0.89
	400	0.10	0.93	0.91	0.94	0.92	0.94	0.92
		0.25	0.94	0.96	0.94	0.96	0.94	0.95
		0.50	0.95	0.98	0.96	0.98	0.95	0.97
		0.75	0.94	0.96	0.94	0.96	0.95	0.96
		0.90	0.91	0.88	0.92	0.90	0.93	0.91
	1000	0.10	0.93	0.91	0.94	0.91	0.95	0.94
		0.25	0.94	0.95	0.96	0.97	0.95	0.96
		0.50	0.95	0.97	0.96	0.97	0.95	0.97
		0.75	0.95	0.96	0.93	0.95	0.96	0.97
		0.90	0.93	0.91	0.93	0.91	0.93	0.92
Model 3	100	0.10	0.92	0.89	0.93	0.89	0.93	0.91
		0.25	0.95	0.96	0.94	0.97	0.95	0.97
		0.50	0.95	0.98	0.95	0.98	0.95	0.98
		0.75	0.97	0.98	0.95	0.98	0.95	0.96
		0.90	0.94	0.91	0.94	0.91	0.94	0.91
	400	0.10	0.95	0.93	0.93	0.91	0.94	0.93
		0.25	0.96	0.97	0.95	0.96	0.96	0.97
		0.50	0.95	0.97	0.94	0.97	0.94	0.96
		0.75	0.95	0.97	0.95	0.96	0.96	0.97
		0.90	0.94	0.91	0.93	0.92	0.95	0.93
	1000	0.10	0.94	0.93	0.94	0.93	0.94	0.93
		0.25	0.95	0.96	0.95	0.97	0.96	0.97
		0.50	0.94	0.96	0.97	0.98	0.96	0.97
		0.75	0.96	0.97	0.95	0.96	0.95	0.97
		0.90	0.93	0.92	0.95	0.93	0.95	0.94

Table 6: Comparison of the average length of nominal 95% confidence intervals for the estimated coefficients $\hat{\beta}_0(\tau)$, $\hat{\beta}_1(\tau)$ and $\hat{\beta}_2(\tau)$ under the three models considered at different sample sizes n and quantile levels τ. The bracketed number gives the standard deviation of each average length.

			$\hat{\beta}_0(\tau)$		$\hat{\beta}_1(\tau)$		$\hat{\beta}_2(\tau)$	
	n	τ	OM	NKDE	OM	NKDE	OM	NKDE
Model 1	100	0.10	0.75 (0.160)	0.67 (0.132)	0.11 (0.024)	0.10 (0.020)	0.26 (0.055)	0.23 (0.045)
		0.25	0.56 (0.079)	0.63 (0.095)	0.08 (0.013)	0.09 (0.015)	0.19 (0.026)	0.22 (0.031)
		0.50	0.51 (0.068)	0.62 (0.094)	0.08 (0.011)	0.09 (0.014)	0.18 (0.022)	0.22 (0.030)
		0.75	0.56 (0.087)	0.63 (0.102)	0.08 (0.013)	0.10 (0.015)	0.20 (0.028)	0.22 (0.034)
		0.90	0.75 (0.159)	0.67 (0.129)	0.11 (0.024)	0.10 (0.020)	0.26 (0.054)	0.23 (0.044)
	400	0.10	0.38 (0.046)	0.35 (0.042)	0.06 (0.007)	0.05 (0.006)	0.13 (0.016)	0.12 (0.014)
		0.25	0.28 (0.021)	0.30 (0.024)	0.04 (0.003)	0.05 (0.004)	0.10 (0.007)	0.10 (0.008)
		0.50	0.25 (0.016)	0.29 (0.021)	0.04 (0.002)	0.04 (0.003)	0.09 (0.005)	0.10 (0.007)
		0.75	0.28 (0.020)	0.30 (0.024)	0.04 (0.003)	0.05 (0.004)	0.10 (0.007)	0.10 (0.008)
		0.90	0.38 (0.042)	0.35 (0.039)	0.06 (0.006)	0.05 (0.006)	0.13 (0.014)	0.12 (0.013)
	1000	0.10	0.24 (0.017)	0.23 (0.017)	0.04 (0.003)	0.03 (0.003)	0.08 (0.006)	0.08 (0.006)
		0.25	0.18 (0.008)	0.19 (0.010)	0.03 (0.001)	0.03 (0.001)	0.06 (0.003)	0.06 (0.003)
		0.50	0.16 (0.006)	0.18 (0.008)	0.02 (0.001)	0.03 (0.001)	0.06 (0.002)	0.06 (0.003)
		0.75	0.18 (0.008)	0.19 (0.009)	0.03 (0.001)	0.03 (0.001)	0.06 (0.003)	0.07 (0.003)
		0.90	0.24 (0.018)	0.23 (0.018)	0.04 (0.003)	0.03 (0.003)	0.08 (0.006)	0.08 (0.006)
Model 2	100	0.10	0.30 (0.069)	0.27 (0.057)	0.05 (0.011)	0.04 (0.009)	0.10 (0.023)	0.09 (0.019)
		0.25	0.22 (0.035)	0.25 (0.041)	0.03 (0.005)	0.04 (0.006)	0.08 (0.011)	0.09 (0.014)
		0.50	0.20 (0.028)	0.24 (0.037)	0.03 (0.004)	0.04 (0.006)	0.07 (0.009)	0.08 (0.012)
		0.75	0.22 (0.034)	0.25 (0.040)	0.03 (0.005)	0.04 (0.006)	0.08 (0.011)	0.09 (0.013)
		0.90	0.30 (0.065)	0.27 (0.053)	0.05 (0.010)	0.04 (0.008)	0.10 (0.022)	0.09 (0.018)
	400	0.10	0.15 (0.019)	0.14 (0.018)	0.02 (0.003)	0.02 (0.003)	0.05 (0.006)	0.05 (0.006)
		0.25	0.11 (0.009)	0.12 (0.010)	0.02 (0.001)	0.02 (0.001)	0.04 (0.003)	0.04 (0.003)
		0.50	0.10 (0.007)	0.11 (0.009)	0.01 (0.001)	0.02 (0.001)	0.03 (0.002)	0.04 (0.003)
		0.75	0.11 (0.009)	0.12 (0.010)	0.02 (0.001)	0.02 (0.001)	0.04 (0.003)	0.04 (0.003)
		0.90	0.15 (0.018)	0.14 (0.018)	0.02 (0.003)	0.02 (0.003)	0.05 (0.006)	0.05 (0.006)
	1000	0.10	0.10 (0.008)	0.09 (0.007)	0.01 (0.001)	0.01 (0.001)	0.03 (0.003)	0.03 (0.003)
		0.25	0.07 (0.003)	0.07 (0.004)	0.01 (0.001)	0.01 (0.001)	0.02 (0.001)	0.03 (0.001)
		0.50	0.06 (0.003)	0.07 (0.003)	0.01 (0.001)	0.01 (0.001)	0.02 (0.001)	0.02 (0.001)
		0.75	0.07 (0.003)	0.07 (0.004)	0.01 (0.001)	0.01 (0.001)	0.02 (0.001)	0.03 (0.001)
		0.90	0.10 (0.008)	0.09 (0.008)	0.01 (0.001)	0.01 (0.001)	0.03 (0.003)	0.03 (0.003)
Model 3	100	0.10	0.67 (0.120)	0.61 (0.103)	0.10 (0.018)	0.09 (0.015)	0.23 (0.040)	0.21 (0.034)
		0.25	0.52 (0.060)	0.58 (0.070)	0.08 (0.009)	0.09 (0.011)	0.18 (0.018)	0.20 (0.022)
		0.50	0.48 (0.048)	0.57 (0.067)	0.07 (0.008)	0.09 (0.010)	0.17 (0.016)	0.20 (0.022)
		0.75	0.52 (0.064)	0.58 (0.074)	0.08 (0.010)	0.09 (0.011)	0.18 (0.019)	0.20 (0.023)
		0.90	0.67 (0.122)	0.60 (0.099)	0.10 (0.019)	0.09 (0.015)	0.23 (0.040)	0.21 (0.032)
	400	0.10	0.33 (0.030)	0.31 (0.030)	0.05 (0.005)	0.05 (0.005)	0.12 (0.010)	0.11 (0.010)
		0.25	0.26 (0.015)	0.28 (0.018)	0.04 (0.002)	0.04 (0.003)	0.09 (0.005)	0.10 (0.006)
		0.50	0.24 (0.012)	0.26 (0.016)	0.04 (0.002)	0.04 (0.002)	0.08 (0.004)	0.09 (0.005)
		0.75	0.26 (0.015)	0.28 (0.018)	0.04 (0.002)	0.04 (0.003)	0.09 (0.005)	0.10 (0.006)
		0.90	0.33 (0.030)	0.31 (0.030)	0.05 (0.005)	0.05 (0.005)	0.12 (0.010)	0.11 (0.010)
	1000	0.10	0.21 (0.012)	0.20 (0.013)	0.03 (0.002)	0.03 (0.002)	0.07 (0.004)	0.07 (0.004)
		0.25	0.16 (0.006)	0.17 (0.008)	0.02 (0.001)	0.03 (0.001)	0.06 (0.002)	0.06 (0.003)
		0.50	0.15 (0.005)	0.16 (0.006)	0.02 (0.001)	0.02 (0.001)	0.05 (0.002)	0.06 (0.002)
		0.75	0.16 (0.006)	0.17 (0.007)	0.02 (0.001)	0.03 (0.001)	0.06 (0.002)	0.06 (0.002)
		0.90	0.21 (0.012)	0.20 (0.013)	0.03 (0.002)	0.03 (0.002)	0.07 (0.004)	0.07 (0.005)

We consider the monthly not seasonally adjusted total CPI (1982–1984 = 100) of the United States, available at the U.S. Bureau of Labor Statistics web (`https://www.bls.gov/cpi/data.htm`). The data consists of $n = 241$ monthly observations for the time series from September 2003 to September 2023. This data was previously analyzed by various authors, for example, by Li and Zhao (2019), in the case of two monthly CPI measures: the seasonally adjusted total CPI and the core CPI, which excludes food and energy. The monthly inflation rate Y_i is the percentage change of CPI from month i to $i-1$:

$$Y_i = \frac{\text{CPI}_i}{\text{CPI}_{i-1}} - 1,$$

where CPI_i is the CPI at month i.

Figure 5(a) displays the monthly total inflation rate Y_i. The graph shows that the volatility in the dataset varies over time, with some periods being more volatile than others. Specifically, the period around 2008 exhibits significantly higher volatility than any other time frame considered in this study. However, after the 2008 crisis, the monthly total inflation rate appeared to exhibit less volatility, which is similar to the earlier 2004. Similar to Li and Zhao (2019), we calculate the 60-month moving average trend (the thick full curve in Figure 5(a)) to visualize the general behavior of the data set. Thus, this 60-month moving average trend shows that the time series appears stationary around a constant mean. In fact, as the stationarity test of Kwiatkowski et al. (1992) (`R` command `kpss.test` in package `tseries`) has a p-value of 0.1, the null hypothesis of stationarity is not rejected at a level of significance 0.05. Also, the unit root test in Perron (1988) (`R` command `pp.test`) indicates that the time series does not have a unit root (p-value=0.01), and then it is stationary. Thus, based on the observed behavior of changing time series variability and the fact that the time series is stationary, it is possible to suggest that an appropriate model for this case should have a volatility structure that varies with time, such as the autoregressive model with ARCH errors (AR-ARCH).

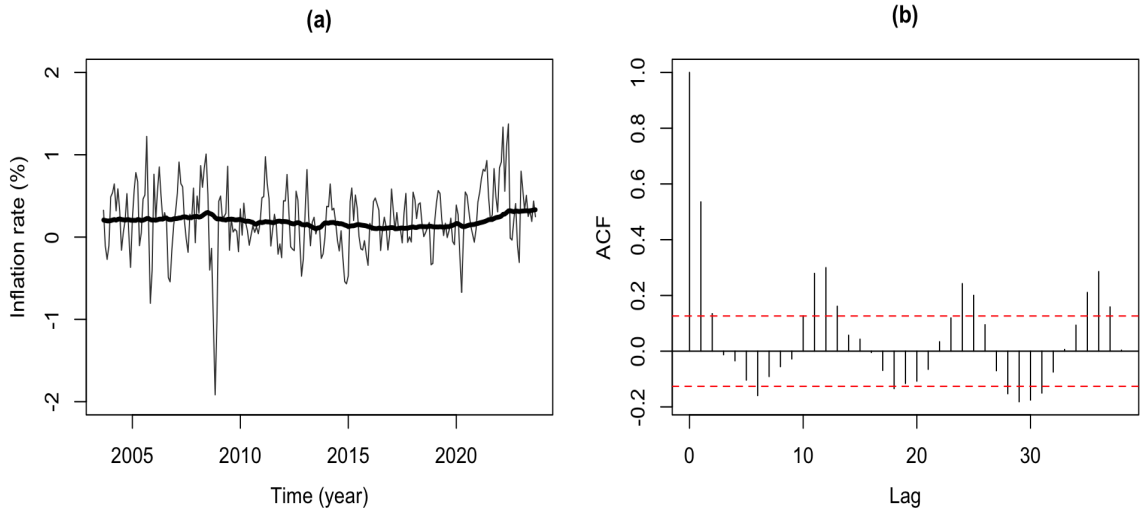


Figure 5: (a) Monthly total inflation rate of the United States, and 60-month moving average trend (▬) during September 2003-September 2023, and (b) The estimated sample autocorrelation function for the monthly total inflation rate.

To visualize the serial correlation and the no presence of a unit root in the monthly total inflation rate, we use Figure 5(b). The graph exhibits the estimated sample autocorrelation function (ACF), where the dashed red lines show the 95% confidence interval for these estimated sample ACF. The ACF shows a fast decaying of the autocorrelations, which confirms that the time series does not have a unit root. Also, the seasonal autocorrelation relationships are shown quite prominently, as there are significant correlations at lags 12, 24, 36, and so on, which suggests the possibility of including a seasonal component in the model.

We use the traditional time series approach to find an adequate model for our variable of interest, as we are more interested in estimating the quantity $f_Z(F_Z^{-1}(\tau))$ present in the asymptotic variance of $\hat{\beta}(\tau)$ than in finding the best model for the inflation rate to forecast. Thus, using the Akaike information criterion, we select lags 1, 2, and 12 of the monthly total inflation rate as the explanatory variables that generate an appropriate model. For $1 \leq i \leq n$, we suggest that the following autoregressive regression model with nonparametric ARCH errors can be an adequate model for the monthly total inflation rate data:

$$Y_i = \beta_0 + \beta_1 Y_{i-1} + \beta_2 Y_{i-2} + \beta_3 Y_{i-12} + Z_i \quad \text{with} \quad Z_i \sim \text{ARCH model in (1)}. \tag{30}$$

Having specified the monthly total inflation rate model, we present estimation results for the vector $\beta = (\beta_0, \beta_1, \beta_2, \beta_3)^T$ using the quantile regression estimator in (19) (`R` command `qr` in package `quantreg`) together with their 95% confidence intervals using the two methods described in Subsection 3.3. To compute the confidence interval, for any fixed τ, of each $\hat{\beta}_j(\tau)$, $j = 0, 1, 2, 3$, we need to estimate its limiting variance, $\hat{\Gamma}(\hat{\beta}_j(\tau))$ which depends on the reciprocal of the density function of the model error terms, $1/f_Z^2(F_Z^{-1}(\tau))$. For comparison, we estimate $f_Z(F_Z^{-1}(\tau))$ using the two methods: OM method and NKDE method.

In the first method (OM), for fixed value $\tau \in (0, 1)$, based on the centered estimated residuals of the model (30), we estimate $f_Z(F_Z^{-1}(\tau))$ using the proposed density estimator $\hat{f}_{\hat{Z}}(\hat{F}_{\hat{Z}}^{-1}(\tau))$ in (6) and thus we get the estimation of the limiting variance $\hat{\Gamma}(\hat{\beta}_j(\tau))$, for $j = 0, 1, 2, 3$. In the second method (NKDE), we use the nonparametric kernel density estimator $\tilde{f}_{\hat{Z}}(\hat{F}_{\hat{Z}}^{-1}(\tau))$ in (2) to estimate the quantity $f_Z(F_Z^{-1}(\tau))$ and in the same way get an estimation of the limiting variance.

Table 7: Comparison of length of the estimated 95% confidence intervals for the estimated coefficients $\hat{\beta}_0(\tau)$, $\hat{\beta}_1(\tau)$, $\hat{\beta}_2(\tau)$ and $\hat{\beta}_3(\tau)$ in the model (30), by utilizing the proposed estimator $\hat{f}_{\hat{Z}}$ (OM method) and the usual nonparametric kernel density estimator $\tilde{f}_{\hat{Z}}$ (NKDE method) for seven quantiles τ.

	Length C.I of $\hat{\beta}_0(\tau)$		Length C.I of $\hat{\beta}_1(\tau)$		Length C.I of $\hat{\beta}_2(\tau)$		Length C.I of $\hat{\beta}_3(\tau)$	
τ	OM	NKDE	OM	NKDE	OM	NKDE	OM	NKDE
0.05	0.19	0.16	0.47	0.40	0.45	0.39	0.40	0.34
0.10	0.17	0.16	0.41	0.40	0.40	0.38	0.35	0.34
0.25	0.13	0.13	0.32	0.33	0.31	0.32	0.27	0.28
0.50	0.12	0.13	0.31	0.32	0.30	0.31	0.26	0.27
0.75	0.13	0.14	0.32	0.35	0.31	0.33	0.27	0.29
0.90	0.18	0.17	0.45	0.43	0.44	0.41	0.38	0.36
0.95	0.20	0.17	0.48	0.41	0.47	0.39	0.41	0.35

Table 7 presents the length of the estimated 95% confidence intervals (C.I.) for the estimated coefficients $\hat{\beta}_0(\tau)$,

$\hat{\beta}_1(\tau)$, $\hat{\beta}_2(\tau)$ and $\hat{\beta}_3(\tau)$ in the model (30), using the two methods: the proposed estimator $\hat{f}_{\hat{Z}}$ (OM method) and the nonparametric kernel density estimator $\tilde{f}_{\hat{Z}}$ (NKDE method). Seven quantiles were considered: 0.05, 0.10, 0.25, 0.50, 0.75, 0.90 and 0.95. It can be seen that for quantiles τ close to 0.5, specifically for τ=0.25, 0.50, and 0.75, both methods show similar confidence interval lengths. However, for most cases, our method (OM) produces slightly shorter lengths of confidence intervals than the NKDE method. For values of τ less than 0.1 and greater than 0.9, the OM method produces wider confidence intervals. As mentioned in the last Subsection 3.3, the behavior in the length of C.I. associated with small and high levels of τ is due to the positive bias in the estimator $\tilde{f}_{\hat{Z}}$ (NKDE method). To illustrate the behavior of the two methods in estimating the confidence intervals of the estimated coefficients in more detail, Figure (6) shows the plots of these confidence intervals for a sequence of different quantiles τ.

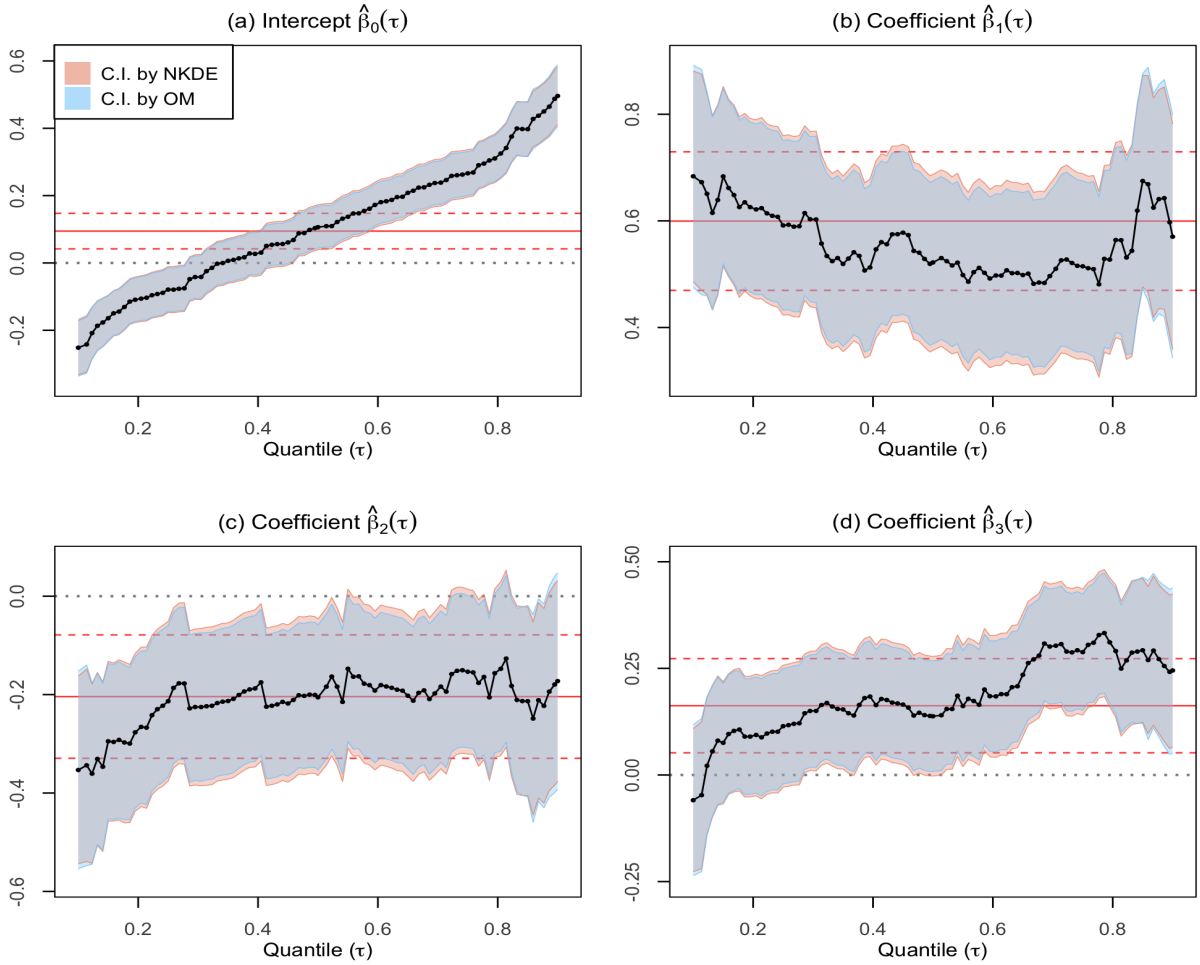


Figure 6: Estimated 95% confidence intervals (C.I) for $\hat{\beta}(\tau)$ in Model (30) using two methods to compute the asymptotic variance: the first method, using $\tilde{f}_{\hat{Z}}$ (NKDE method), is indicated by the shaded area marked in salmon color ( ), and the second method, the proposed method (OM) using $\hat{f}_{\hat{Z}}$, is indicated by the shaded area in blue ( ). The red line on each plot represents the OLS estimate of the mean effect, with two red dashed lines again representing a 95% C.I for this coefficient. The gray dotted line ($\cdots$) indicates $\beta = 0$, the null effect. (a) C.I for intercept $\hat{\beta}_0(\cdot)$; (b) C.I for coefficient $\hat{\beta}_1(\cdot)$; (c) C.I for coefficient $\hat{\beta}_2(\cdot)$; (d) C.I for coefficient $\hat{\beta}_3(\cdot)$.

Figure 6 summarizes the quantile regression results for the application to the monthly total inflation rate. Each plot displays one estimated coefficient from the quantile autoregressive model with nonparametric ARCH errors in (30). We consider a sequence of 100 equally spaced quantile levels τ between 0.05 and 0.95. The black dots represent the estimated coefficients, $\{\hat{\beta}_j(\tau) : j = 0, 1, 2, 3.\}$, for the quantiles indicated on the x-axis, with a solid black line connecting them. The shaded area marked in blue represents the estimated 95% confidence intervals using the proposed method (OM), while the shaded area marked in salmon color represents the estimated 95% confidence intervals using the NKDE method. Additionally, a red line is superimposed on each plot, representing the common ordinary least squares (OLS) estimate of the mean effect, with two red dashed lines again representing the OLS 95% confidence interval for this coefficient. The gray dotted line $(\cdots)$ indicates $\beta_j = 0$, the null effect. We corroborate here the fact that for quantiles close to 0.5 the proposed method (OM) results in narrower confidence intervals than the NKDE method for $\hat{\beta}_1(\cdot)$, $\hat{\beta}_2(\cdot)$ and $\hat{\beta}_3(\cdot)$. In contrast, for small and high quantiles, our method (OM) exhibits wider confidence intervals compared to the NKDE method. In this case, it is better to use our proposed density estimator $\hat{f}_{\hat{Z}}(z)$ in (6) (OM method), based on the estimated residuals of the model, to estimate the asymptotic variance of $\hat{\beta}_j(\tau)$'s in the quantile regression model compared with the NKDE method.

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Appendix

A Proofs

Write $\|Z\|^2 = \mathbb{E}\left(Z^2\right)$ for a random variable Z. Throughout, denote by $\phi\left(z;\sigma^2\right)$ the normal density of $N\left(0,\sigma^2\right)$, and denote the partial derivative by $\dot{\phi}_{\sigma^2}\left(z;\sigma^2\right) = \partial\phi\left(z;\sigma^2\right)/\partial\sigma^2$. Also, $c_1, c_2, \ldots$, are generic constants that may vary from places to places, and $\mathcal{Z}$ is any bounded interval with length denoted by $|\mathcal{Z}|$. For convenience, throughout we write $K_{h_n}(u) = K\left(u/h_n\right)/h_n$.

Lemma 1 below is from Zhao and Hernandez (2023) and the proof is omitted here.

Lemma 1. *For any differentiable random function $H(y)$,*

$$\sup_{y\in\mathcal{Y}}|H(y)| = O_p\left\{\inf_{y\in\mathcal{Y}}\|H(y)\| + \sup_{y\in\mathcal{Y}}\|H'(y)\|\right\}. \tag{31}$$

Lemma 2. *Let $g(\cdot;\cdot)$ and $h(\cdot)$ be functions such that $\mathbb{E}h(e_0) = 0$, $\mathbb{E}h^2(e_0) < \infty$, $\sup_{x\in\mathcal{X}}\mathbb{E}[g^2(z;Z_0) + \dot{g}^2(z;Z_0)] < c$. Define*

$$\Lambda_n(z) = \sum_{j=1}^{n} g(z;Z_{j-1})h(e_j).$$

Then $\sup_{z\in\mathcal{Z}}|\Lambda_n(z)| = O_p(n^{1/2})$.

Proof. Recall the sigma filed $\mathcal{F}_i$ in Assumption 2. By causality, $\mathbb{E}[g(z;X_{j-1})h(e_j)|\mathcal{F}_{j-1}] = g(z;Z_{j-1})\mathbb{E}[h(e_j)|\mathcal{F}_{j-1}] = 0$ via the i.i.d. assumption of $\{e_j\}_j$. Thus, for each fixed z, $\{g(z;Z_{j-1})h(e_j)\}_i$ form martingale differences. By orthogonality of martingale differences,

$$\begin{aligned}\mathbb{E}\Lambda_n(z)^2 = \sum_{j=1}^{n}\mathbb{E}[g^2(z;Z_{j-1})h^2(e_j)] &= \sum_{j=1}^{n}\mathbb{E}g^2(z;Z_{j-1})\mathbb{E}h^2(e_j)\\ &\le n\mathbb{E}h^2(e_0)\sup_{z\in\mathcal{Z}}\mathbb{E}[g^2(z;Z_0).\end{aligned}$$

Similarly, for the derivative $\Lambda_n'(z) = \sum_{i=1}^{n}\dot{g}(z;Z_{j-1})h(e_j)$, by the same argument, we have

$$\mathbb{E}\Lambda_n'(z)^2 \le n\mathbb{E}h^2(e_0)\sup_{z\in\mathcal{Z}}\mathbb{E}[\dot{g}^2(z;Z_0).$$

The desired result then follows from Lemma 1 under the specified condition. □

Lemma 3. *Let function $g(\cdot;\cdot)$ satisfy $\sup_{z\in\mathcal{Z}}\mathbb{E}[g^2(z;Z_0)+\dot{g}^2(z;Z_0)]<c$. Define*

$$\Xi_{j,n}(z)=\frac{1}{n}\sum_{i=1}^{n}K_{h_n}(Z_{i-1}-Z_{j-1})g(z;Z_{i-1}).$$

Then

$$\sup_{1\le j\le n,z\in\mathcal{Z}}|\Xi_{j,n}(z)-f_Z(Z_{j-1})g(z;Z_{j-1})|=O_p\left(h_n^2+\frac{\log n}{\sqrt{nh_n}}\right).$$

Proof. Define

$$\Xi_n(x,z)=\frac{1}{n}\sum_{i=1}^{n}K_{h_n}(Z_{i-1}-x)g(z;Z_{i-1}).$$

For example, the special case of $g(\cdot;\cdot)=1$ is just the nonparametric kernel density estimate. Following the same lines of proofs in Hansen (2008), we can show that the following approximation holds

$$\Xi_n(x,z)=f_Z(z)g(z;x)+O_n\left(h_n^2+\frac{\log n}{\sqrt{nh_n}}\right),\quad \text{uniformly over } z\in\mathcal{Z},\ |x|=O(n). \tag{32}$$

On the other hand, from the condition $\mathbb{E}|Z_0|^s<\infty$ for some $s>2$, we have $\max_j|Z_j|=O_p(\sqrt{n})$, which together with (32) gives the desired result. □

Proof of Theorem 1. Under Assumptions 1 and 2, by Hansen (2008), we have

$$\sup_{|z|=O(n)}\left|\hat{\sigma}^2(z)-\sigma^2(z)\right|=O_p\left(h_n^2+\frac{\log n}{\sqrt{nh_n}}\right) \tag{33}$$

The expression in (33) provides a uniform error bound for the estimate $\hat{\sigma}^2(\cdot)$. Define

$$\delta_n:=\sup_{1\le i\le n}\left|\hat{\sigma}^2\left(Z_{i-1}\right)-\sigma^2\left(Z_{i-1}\right)\right|^2. \tag{34}$$

As mentioned in the proof of Lemma 3, we have $\max_i|Z_i|=O_p(\sqrt{n})$, which together with (33) implies that

$$\delta_n=O_p\left(h_n^4+\frac{(\log n)^2}{nh_n}\right)=o_p(n^{-1/2}),\quad \text{under bandwidth condition in Assumption 1.} \tag{35}$$

Note that, the first and second order derivatives of $\phi\left(z;\sigma^2\right)$ with respect to $\sigma^2>\sigma_0^2$ are all uniformly bounded in $z\in\mathbb{R}$. Applying Taylor's expansion of $\phi\left(z;\hat{\sigma}^2\left(Z_{i-1}\right)\right)$ at $\sigma^2\left(Z_{i-1}\right)$ and using (34) and (35), we can obtain

$$\hat{f}_Z(z)=\frac{1}{n}\sum_{i=1}^{n}\phi\left(z;\sigma^2\left(Z_{i-1}\right)\right)+I_n(z)+o_p(n^{-1/2}), \tag{36}$$

where

$$I_n(z) = \frac{1}{n}\sum_{i=1}^{n}\left[\hat{\sigma}^2\left(Z_{i-1}\right) - \sigma^2\left(Z_{i-1}\right)\right]\dot{\phi}\left(z;\sigma^2\left(Z_{i-1}\right)\right) \quad \text{with} \quad \dot{\phi}\left(z;\sigma^2\right) = \frac{\partial\phi(z;\sigma^2)}{\partial\sigma^2}. \tag{37}$$

For the nonparametric estimate $\hat{\sigma}^2(z)$ in (7), recall the notation $K_{h_n}(u) = K\left(u/h_n\right)/h_n$, we can rewrite

$$\hat{\sigma}^2(z) = \frac{\sum_{j=1}^{n} K_{h_n}\left(z - Z_{j-1}\right) Z_j^2}{n\tilde{f}_Z(z)} \quad \text{where} \quad \tilde{f}_Z(z) = \frac{1}{n}\sum_{j=1}^{n} K_{h_n}\left(z - Z_{j-1}\right). \tag{38}$$

Using $Z_j = \sigma\left(Z_{j-1}\right)e_j$, we can obtain

$$\hat{\sigma}^2(z) - \sigma^2(z) = B_n(z) + \frac{\sum_{j=1}^{n} K_{h_n}\left(z - Z_{j-1}\right)\sigma^2(z)\left[e_j^2 - 1\right]}{n\tilde{f}_Z(z)} \tag{39}$$

with

$$B_n(z) = \frac{\sum_{j=1}^{n} K_{h_n}\left(z - Z_{j-1}\right)\left[\sigma^2\left(Z_{j-1}\right) - \sigma^2(z)\right]e_j^2}{n\tilde{f}_Z(z)}. \tag{40}$$

The bias term $B_n(z)$ satisfies $\sup_z |B_n(z)| = O_p(h_n^2)$. Thus, by (39) and the uniform boundedness of $\dot{\phi}\left(z;\sigma\left(Z_{i-1}\right)\right)$, we have:

$$I_n(z) = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{K_{h_n}\left(Z_{i-1} - Z_{j-1}\right)\sigma^2(Z_{i-1})\left[e_j^2 - 1\right]}{n\tilde{f}_Z(Z_{i-1})}\dot{\phi}\left(z;\sigma^2\left(Z_{i-1}\right)\right) + O_p\left(h_n^2\right). \tag{41}$$

For $\tilde{f}_Z(z)$ in (38), by property [Hansen (2008)] of nonparametric kernel density estimator,

$$\tilde{f}_Z(z) = f_Z(z) + O_p\left(h_n^2 + \sqrt{\frac{\log n}{nh_n}}\right), \quad \text{uniformly in } z. \tag{42}$$

Here $f_Z(z)$ is the density of Z_i. Substituting this in (41), we can express $I_n(z)$ as

$$I_n(z) = \left[1 + O_p\Big(h_n^2 + \sqrt{\frac{\log n}{nh_n}}\Big)\right]\tilde{I}_n(z) + O_p\left(h_n^2\right), \tag{43}$$

where

$$\tilde{I}_n(z) = \frac{1}{n}\sum_{j=1}^{n}\left[\frac{1}{n}\sum_{i=1}^{n}\frac{K_{h_n}\left(Z_{i-1} - Z_{j-1}\right)\dot{\phi}\left(z;\sigma^2\left(Z_{i-1}\right)\right)\sigma^2(Z_{i-1})}{f_Z\left(Z_{i-1}\right)}\right](e_j^2 - 1).$$

By Lemma 3 and the fact $\sum_{j=1}^{n}(e_j^2 - 1) = O_p(\sqrt{n})$, we can further approximate $\tilde{I}_n(z)$ as

$$\tilde{I}_n(z) = \tilde{I}_n^*(z) + O_p\left(h_n^2 + \frac{\log n}{n\sqrt{h_n}}\right), \quad \text{where} \quad \tilde{I}_n^*(z) = \frac{1}{n}\sum_{j=1}^{n} \dot{\phi}\left(z; \sigma^2\left(Z_{j-1}\right)\right) \sigma^2(Z_{j-1})(e_j^2 - 1). \tag{44}$$

By Lemma 2, $\sup_{z\in\mathcal{Z}} |\tilde{I}_n^*(z)| = O_p(n^{-1/2})$. Combining this latter fact with (43), (44), and the bandwidth condition in Assumption 1, we have

$$I_n(z) = \frac{1}{n}\sum_{j=1}^{n} \dot{\phi}\left(z; \sigma^2\left(Z_{j-1}\right)\right) \sigma^2(Z_{j-1})(e_j^2 - 1) + o_p(n^{-1/2}). \tag{45}$$

From (36), (45) and the fact $f_Z(z) = \mathbb{E}\phi(z; \sigma^2(Z_{i-1}))$, we can obtain

$$\hat{f}_Z(z) - f_Z(z) = \frac{1}{n}\sum_{i=1}^{n} \xi_i(z) + o_p(n^{-1/2}), \tag{46}$$

where $\xi_i(z)$ is defined as in (10). It is well known that central limit theorems hold under the mixing condition in Assumption 2. The finite conditional convergence then follows from the Cramér-Wold device, and the tightness clearly holds from the differentiability of $\xi_i(z)$. This completes the proof.

□

Proof of Proposition 1. From the proof of Theorem 1, $\hat{\sigma}^2(Z_{i-1}) = \sigma^2(Z_{i-1}) + o_p(1)$, uniformly over i. Thus, from the uniform consistency of $\hat{f}_Z(z)$ and the boundedness of $\phi(z;\sigma^2)$ and $\dot{\phi}(z;\sigma^2)$, we can easily obtain

$$\begin{aligned}
\hat{\xi}_i(z) &= [\phi(z;\sigma^2(Z_{i-1}) - f_Z(z) + o_p(1)] + [\dot{\phi}(z;\sigma^2(Z_{i-1})) + o_p(1)][Z_i^2 - \sigma^2(Z_{i-1}) + o_p(1)] \\
&= \xi_i(z) + o_p(1) + o_p(1)[Z_i^2 + \sigma^2(Z_{i-1})],
\end{aligned}$$

uniformly over $1 \le i \le n$ and $z \in \mathcal{Z}$. The result then easily follows from the law of large numbers. □

Proof of Corollary 1. In (15), for Gaussian kernel $K(u)$, $\int_{\mathbb{R}} u^2 K(u)du = 1$. Let $Z = (Z_1, ..., Z_{i-1}, ..., Z_{n-1})$. Now, to compute $f_Z''(z)$, under the nonparametric autoregressive conditional heteroscedasticity model in (1), note that by (4) the density function f_Z can be written as

$$f_Z(z) = \mathbb{E}\left[\frac{1}{\sigma(Z)}\phi\left(\frac{z}{\sigma(Z)}; 1\right)\right], \tag{47}$$

where $\phi(\cdot; 1)$ denotes the standard normal density of $N(0, 1)$. Thus,

$$f_Z''(z) = \mathbb{E}\left[\frac{1}{\sigma(Z)}\frac{\partial^2}{\partial z^2}\phi\left(\frac{z}{\sigma(Z)}; 1\right)\right], \tag{48}$$

Elementary calculations show

$$\frac{\partial^2}{\partial z^2}\phi\left(\frac{z}{\sigma(Z)};1\right)=\frac{1}{\sigma^2(Z)}\left(\frac{z^2}{\sigma^2(Z)}-1\right)\phi\left(\frac{z}{\sigma(Z)};1\right), \tag{49}$$

completing the proof.

□