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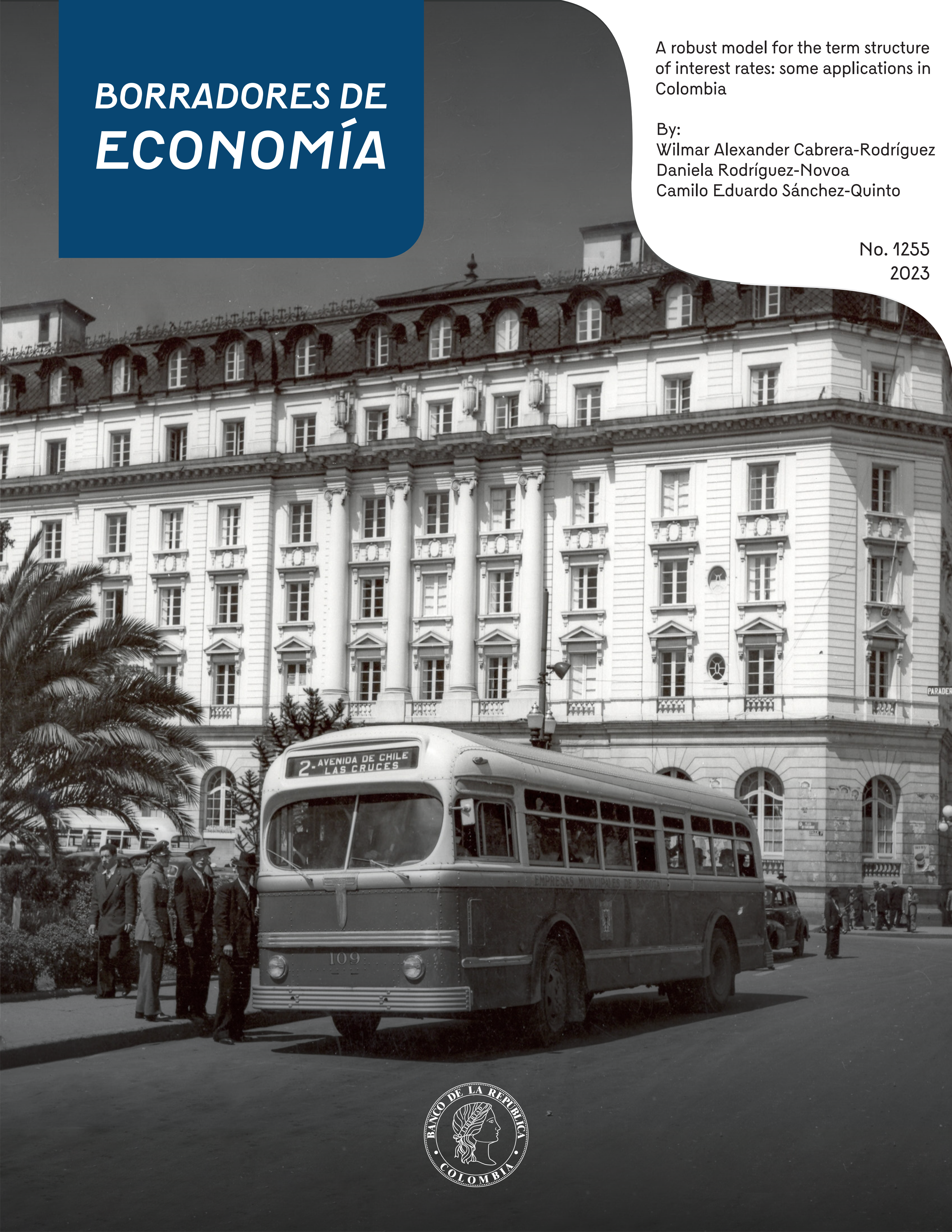

A robust model for the term structure of interest rates: some applications in Colombia

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Abstract

This document presents a Gaussian Affine Term Structure Model (GATSM) of the zero-coupon public debt curve issued locally by the Colombian Government, adopting the methodological approach of Hamilton and Wu (2012) to solve the problems of identification and instability in the estimation of this family of models. Two empirical exercises are presented to highlight the relevance of this methodological approach. The first combines the GATSM structure with a Bayesian Averaging of Classical Estimates (BACE) approach to forecast the yield curve given a set of macroeconomic variables, thus offering a practical way to link a macroeconomic scenario to financial prices in a stress testing exercise. In particular, the document presents the connection with the Systemic Stress Model (SYSMO) of the Financial Stability Department of the Central Bank of Colombia. The second evaluates the effect of monetary policy surprises on sovereign bond yields on a comprehensive set of maturities in a parsimonious way allowed by the GATSM structure. We found an almost immediate, complete, and significant pass-through on the short end of the yield curve. These empirical applications reflect the flexibility of this approach as a tool to address studies that deepen the understanding of the dynamics of yield curves and macroeconomics, the valuation of financial instruments, and financial stability.

Keywords: Affine term structure models, Bond Interest Rates, Financial Markets and the Macroeconomy, Monetary Policy.

JEL Classification: E43, G12, E44, E52

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Un modelo robusto para la estructura a término de las tasas de interés: algunas aplicaciones en Colombia

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Resumen

Este documento presenta un Modelo Afín Gaussiano para la Estructura a Plazos (GATSM, por sus siglas en inglés) de la curva cero cupón de los títulos de deuda pública emitidos localmente por el Gobierno colombiano, adoptando el enfoque metodológico de Hamilton y Wu (2012) para resolver los problemas de identificación e inestabilidad en la estimación de esta familia de modelos. Se presentan dos ejercicios empíricos para resaltar la relevancia de este enfoque metodológico. El primero combina la estructura GATSM con un enfoque de *Bayesian Averaging of Classical Estimates* (BACE) para predecir la curva de rendimientos dado un conjunto de variables macroeconómicas, ofreciendo así una forma práctica de vincular un escenario macroeconómico a los precios financieros en un ejercicio de pruebas de estrés. En particular, el documento presenta la conexión con el modelo de estrés sistémico (SYSMO) del Departamento de Estabilidad Financiera del Banco de la República de Colombia. El segundo ejercicio evalúa el efecto de las sorpresas de política monetaria sobre los rendimientos de los bonos soberanos en un conjunto amplio de vencimientos de una manera parsimoniosa permitida por la estructura del GATSM. Encontramos una transmisión casi inmediata, completa y significativa en el extremo corto de la curva de rendimientos. Estas aplicaciones empíricas reflejan la flexibilidad de este enfoque como herramienta para abordar estudios que profundizan en la relación entre las curvas de rendimiento y la macroeconomía, la valoración de los instrumentos financieros y la estabilidad financiera.

Palabras clave: Modelos afín de la estructura a término de las tasas de interés, tasas de interés de los bonos, mercados financieros y macroeconomía, política monetaria.

Clasificación JEL: E43, G12, E44, E52

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1 Introduction

After the mathematical formalization of Gaussian Models with an Affine Structure[7] of the Yield Curve[8] (GATSM) conducted by Duffie and Kan (1996), Dai and Singleton (2000), and Duffee (2002), among other authors, the GATSM has been positioned as a standard method to understand the dynamics of the sovereign debt yield curve as a function of a reduced set of factors, with multiple empirical applications in the international literature[9] (Ang and Piazzesi, 2003; Diebold, Piazzesi and Rudebusch, 2005; Duffee, 2006; Diez de los Rios, 2009; Cochrane and Piazzesi, 2009; Rudebush and Wu, 2008; Hodrick and Tomunem, 2018).

The attractiveness of these models lies in their mathematical tractability, the inclusion of non-arbitrage restrictions[10] that allows an approximation of the entire yield curve by estimating a small number of observed returns and a risk premium that changes over time[11], in line with the empirical evidence against the yield curve expectations hypothesis[12]. However, only some authors have addressed the instability and identification problems due to the large number of parameters, the non-linear relationships in these models, and the high persistence of bond yields. The approximation of Hamilton and Wu (2012) used in this work is a general estimation proposal without such weaknesses.

In Colombia, the applications of GATSM have focused on explaining the yield curve using macroeconomic variables and the Fed funds rate and on calculating the Break-Even Inflation[13] (BEI) (Melo-Velandia and Granados-Castro, 2012; Melo-Velandia and Moreno-Gutiérrez, 2010, Espinosa-Torres et al., 2015, Cuadros, 2015, and Gómez,2016). However, the local literature on asset pricing is scarce. This document embodies previous local literature by offering an updated estimation of these models with an estimation method that fixes the identification and optimization weaknesses. This parsimonious model exhibits an adequate fit in and out of the sample across the entire yield curve.

Furthermore, to highlight the relevance and flexibility embodied in the GATSM, we carry out two application exercises contributing to the empirical literature on this topic. First, following Gross and Poblacion (2019) and Sala-I-Martin et al. (2004) we employ a Bayesian Averaging of Classical Estimates (BACE) to forecast the Colombian nominal sovereign bond (TES) yields for specific maturities given a set of macroeconomic variables. Thereafter, we use the GATSM to build the whole yield curve using at least a combination of three projected yields. We found that this two-step methodology improves the forecast accuracy out of the sample since it shows the lowest forecast error when comparing to the results obtained using solely the GATSM. Specifically, the forecast error measures reduce approximately by 3.1% and 73% in the 3-month and in the 6-month horizon, respectively. Additionally, the curve structure obtained is consistent with different macroeconomic scenarios implying that this methodology allows to predict the yield curve even under a 'tail-risk' scenario. In this regard, we intend to contribute to the

[7] These models assume that the yield curve depends linearly on a set of factors that may be latent or observed.
[8] The yield curve refers to the term structure of interest rates for a given bond market. For more detail on the main concepts surrounding this market, see Appendix 1.
[9] Understanding these markets provides information on the expected dynamics of variables such as inflation and the probability of a recession event. In addition, it is a tool for studying the transmission of monetary policy decisions and is the basis for the valuation of the different financial instruments of the economy.
[10] The non-arbitrage condition implies the absence of any way to obtain additional profits without taking risks of loss.
[11] The risk premium is an investor's compensation for acquiring a risky asset. In the case of nominal public debt bonds, premiums are due mainly to maturity, liquidity, and inflationary risk.
[12] This set of hypotheses postulates that the long-term interest rate is determined solely by current short-term interest rates and their expected value in the future.
[13] Difference between nominal bond rates and inflation-indexed bond rates. In the literature, it is used to break down the dynamics of bonds into inflation expectations, inflation risk premium, and liquidity premium.

market risk satellite model on a fully fledge solvency stress testing as the one employed at the Central Bank of Colombia[14].

The second exercise explores the impact of monetary policy surprises on the yield curve in nominal and real terms using a GATSM adjusted for each market. For this, policy shocks are identified as the difference between the one-month interbank lending rate (IBR for one month) that is observed just before monetary policy decisions and the policy interest rate established by the Board of Directors of Banco de la República (JDBR) in their meeting, following an identification approach similar to Cochrane and Piazzesi (2002). In the second step, using linear projections as in Jordà (2005), the effects of monetary policy surprises on the latent factors extracted from the GATSM are presented. Finally, the GATSM structure provides an empirical tool to expand these results to the entire yield curve. The results suggest that in the event of an unexpected expansionary shock of 20 basis points, the transmission occurs immediately and to a similar magnitude in the COP (nominal) yield curve, with a statistical significance of up to 20 days and a maturity of 5 years. In contrast, the response on the UVR (real) yield curve is about 6 basis points with a statistical significance on the first three days.

This document contains five sections, and this introduction is the first. The second one develops the methodology and the estimation process of the GATSM. The third section characterizes the data. The fourth presents the estimation results for the models and the two empirical applications. Furthermore, in the fifth section, some final considerations are outlined.

2 Methodology

Following the notation in Campbell (2018), the relationship between the price and the yield of a zero-coupon bond with a maturity of *n* years in period *t* and that pays one monetary unit in period *t+n* is:

$$P_{n,t} = \frac{1}{(1+Y_{n,t})^n}$$

In this sense, $Y_{n,t}$ corresponds to the annualized return received by the investor who holds the title until maturity. Taking logarithms, we have:

$$p_{n,t} = -ny_{n,t}$$

2.1 GATSM framework

In the asset pricing literature, the Euler equation relates the marginal utility associated with an economic agent for reducing his consumption to buy an asset and the present value of the expected utility of the payment related to that asset:

$$P_t u^{'}(c_t) = \beta E_t\left[u^{'}(c_{t+1})X_{t+1}\right] \tag{1}$$

Where,

- P_t = asset price in period t.
- $u'(.)$ = first-order derivative of the agent's utility function.
- c_t = consumption in period t.
- X_{t+1} = payment generated by the asset in period *t+1*.
- β = intertemporal discount factor.

[14] See the *Financial Stability Report* - first half of 2023 of the Banco de la República.

From this equation, it is possible to arrive at the standard representation of the price of an asset, based on the present value of its future payments, with a stochastic discount factor M_t[15] as follows:

$$P_t = E_t[M_{t+1}X_{t+1}] \quad (2)$$

$$M_{t+1} = \frac{\beta\, u^{'}(c_{t+1})}{u^{'}(c_t)} \quad (3)$$

On the other hand, when applying this formulation to the price of a zero-coupon bond with maturity n, we find:

$$P_t^n = E_t[M_{t+1}P_{t+1}^{n-1}] \quad (4)$$

Although it is possible to derive a functional form for the stochastic discount factor according to the utility function assumed for the agents that decide to purchase these assets, the GATSM directly assumes a reduced form for M_t (Equation 5). In addition, the model is composed of a risk premium λ_t (Equation 6), a first-order autoregressive dynamic for a set of dynamic factors (Equation 7), as well as an affine structure of the set of latent factors for the short-term interest rate r_t (Equation 8). The discrete-time model is presented as follows[16]:

$$M_{t+1} = \exp\left(-r_t - \frac{1}{2}\lambda_t^{'}\lambda_t - \lambda_t^{'}\epsilon_{t+1}\right) \quad (5)$$

$$\lambda_t = \lambda + \Lambda F_t \quad (6)$$

$$F_{t+1} = c + \rho F_t + \Sigma\epsilon_{t+1} \qquad \epsilon_t \sim N(0, \sigma^2) \quad (7)$$

$$r_t = \delta_0 + \delta_1^{'}F_t \quad (8)$$

The above formulation suggests that these relationships hold for the different maturities of fixed debt securities and, therefore, suggests the existence of a broad set of equations for the yield curve. It is also assumed an affine structure for prices on the latent factors (Equation 9):

$$P_t^n = \exp\,(A_n + B_nF_t) \quad (9)$$

This representation, together with the structure of the model (Equations 4 to 8), allows us to arrive at a representation from which it is possible to find, through the method of indeterminate coefficients, a practical and tractable solution of the model (Equations 10 and 11):

$$P_t^n = E_t[M_{t+1}P_{t+1}^{n-1}]$$

Using Equations 5 and 9:

$$\exp(A_n + B_nF_t) = E_t\left[\exp\left(-r_t - \frac{1}{2}\lambda_t^{'}\lambda_t - \lambda_t^{'}\epsilon_{t+1}\right)\exp\big(A_{n-1} + B_{n-1}(c + \rho F_t + \Sigma\epsilon_{t+1})\big)\right]$$

$$A_{n+1} = A_n + B_n^{'}(c - \Sigma\lambda) + \frac{1}{2}B_n^{'}\Sigma^{'}\Sigma B_n - \delta_0 \quad (10)$$

[15] To implement the GATSM, the assumption of the absence of arbitrage conditions guarantees the existence of a risk-neutral measure from which any asset in the economy can be valued, including bonds. Being ξ_{t+1} the function that converts the risk-neutral measure into the measure of the data-generating process, the GATSM is developed by assuming the following relationship:

$$M_{t+1} = \exp\,(r_t) * \xi_{t+1}/\xi_t$$

Where r_t is the interest rate on short-term bonds (Ang and Piazzesi, 2003; Bansal and Yaron, 2004)

[16] The representation of the model in continuous time is in Piazzesi (2005).

$$B'_{n+1} = B'_{n}(\rho - \Sigma\Lambda) - \delta'_1 \quad (11)$$

This specification establishes that it is enough to estimate {c, δ_0, δ_1, λ, Λ, and Σ} to have a complete representation of the yield curve. Additionally, it is possible to change the measurement to a risk-neutral one[17] (Equations 12 to 18) and in this case the parameters of interest are $\{c^Q, \rho^Q, \delta_0, \delta_1, \lambda, \Lambda\ y\ \Sigma\}$.

$$P_t^n = E_t^Q\left[M_{t+1}^Q P_{t+1}^{n-1}\right] \quad (12)$$

$$F_{t+1} = c^Q + \rho^Q F_t + \Sigma\epsilon_{t+1}^Q \qquad \epsilon_t^Q \sim N(0, \sigma^2) \quad (13)$$

$$c^Q = c - \Sigma\lambda \quad (14)$$

$$\rho^Q = \rho - \Sigma\Lambda \quad (15)$$

$$r_t = \delta_0 + \delta'_1 F_t \quad (16)$$

$$\lambda_t = \lambda + \Lambda F_t \quad (17)$$

$$M_{t+1}^Q = \exp(-r_t) \quad (18)$$

2.2 GATSM estimation

Following the methodology proposed by Hamilton and Wu (2012), a set of structural equations is found from the affine structure previously exposed, and its estimation is performed by Ordinary Least Squares (OLS). Then we follow an estimation method that minimizes a log-likelihood function whose set of parameters follows a chi-square distribution (MCSE).

For ease, the yields are used instead of prices:

$$P_t^n = \exp(-y_n n) = \exp(A_n + B_n F_t) \quad (19)$$

$$y_t^n = -\frac{A_n}{n} + \frac{B_n}{n} F_t \quad (20)$$

$$y_t^n = -\overline{A_n} + \overline{B_n} F_t \quad (21)$$

Additionally, we assume that for a set of returns (Y_t^1), the model does not have an error:

$$\begin{bmatrix} Y_t^1 \\ Y_t^2 \end{bmatrix} = \begin{bmatrix} \overline{A_1} \\ \overline{A_2} \end{bmatrix} + \begin{bmatrix} \overline{B_1} \\ \overline{B_2} \end{bmatrix} F_t + \begin{bmatrix} 0 \\ \Sigma_2 \end{bmatrix} \epsilon_t^2 \quad (22)$$

The procedure to find the set of structural equations for $Y_t^1$is:

$$F_t = c + \rho F_{t-1} + \Sigma\epsilon_t$$

pre-multiplying both sides by $\bar{B}_1$, and by $\bar{B}_1^{-1}\bar{B}_1$ in the second term from the right:

$$\bar{B}_1 F_t = \bar{B}_1 c + \bar{B}_1 \rho \bar{B}_1^{-1} \bar{B}_1 F_{t-1} + \bar{B}_1 \Sigma\epsilon_t$$

[17] Bonds are usually valued with a risk-neutral probability measure ***Q***. In this way, their price corresponds to the present value of future payment flows, discounted with a risk-free rate. Its importance also lies in that if measure ***Q*** exists, the considered system does not contemplate or allow arbitrage opportunities (Ang and Piazzesi, 2003; Hamilton and Wu, 2012).

$$\bar{A}_1 + \bar{B}_1 F_t = \bar{A}_1 + \bar{B}_1 c + \bar{B}_1 \rho \bar{B}_1^{-1} \bar{B}_1 F_{t-1} + \bar{B}_1 \Sigma \epsilon_t$$

Using the normalization $c = 0$, $\Sigma = I$, and the system of Equation 22, we obtain:

$$Y_t^1 = \bar{A}_1^* + \phi_{1,1}^* Y_{t-1}^1 + \epsilon_{1,t}^* \tag{23}$$

Following the same procedure for Y_t^2, the structural equation is found:

$$Y_t^2 = \bar{A}_2^* + \phi_{2,1}^* Y_t^1 + \epsilon_{2,t}^* \tag{24}$$

Where,

$$\bar{A}_1^* = \bar{B}_1 c + \bar{A}_1 - \bar{B}_1 \rho \bar{B}_1^{-1} \bar{A}_1 \tag{25}$$

$$\phi_{1,1}^* = \bar{B}_1 \rho \bar{B}_1^{-1} \tag{26}$$

$$\bar{A}_2^* = \bar{B}_2 c + \bar{A}_1 - \bar{B}_2 \bar{B}_1^{-1} \bar{A}_1 \tag{27}$$

$$\phi_{2,1}^* = \bar{B}_2 \bar{B}_1^{-1} \tag{28}$$

$$\begin{bmatrix} \epsilon_{1,t}^* \\ \epsilon_{2,t}^* \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \Omega_1^* & 0 \\ 0 & \Omega_2^* \end{bmatrix} \right) \tag{29}$$

$$\Omega_1^* = \bar{B}_1 \Sigma \Sigma' \bar{B}_1' \tag{30}$$

$$\Omega_2^* = \Sigma_2 \Sigma \Sigma' \Sigma_2' \tag{31}$$

It is worth mentioning that the following restrictions are used, in line with Dai and Singleton (2000), to guarantee the identification of the parameters of interest:

$$\delta_1 \geq 0$$

$$c = 0$$

$$\Sigma = I$$

Furthermore, to solve the identification problems generated by the high persistence of bond yields, this paper follows the proposal of Hamilton and Wu (2012), considering that three estimated returns are assumed without error (Y_t^1; Equation 23) and the following restrictions are added:

- If ρ^Q contains complex roots:

$$\rho^Q = \begin{bmatrix} \rho_{1,1}^Q & 0 & 0 \\ \rho_{2,1}^Q & a & \rho_{2,3}^Q \\ \rho_{3,1}^Q & \rho_{3,2}^Q & a \end{bmatrix} \tag{32}$$

  With $\rho_{2,3}^Q \leq \rho_{3,2}^Q$

- Otherwise:

$$\rho^Q = \begin{bmatrix} \rho_{1,1}^Q & 0 & 0 \\ \rho_{2,1}^Q & \rho_{2,2}^Q & 0 \\ \rho_{3,1}^Q & \rho_{3,2}^Q & \rho_{3,3}^Q \end{bmatrix} \tag{33}$$

  With $\rho_{1,1}^Q \geq \rho_{2,2}^Q \geq \rho_{3,3}^Q$

In verifying the number of parameters of interest and available equations, only one equation (Y_t^2) observed with error should be considered to have a just-identified model when considering three yields without error. Thus, in the first stage, equations 23 and 24 are estimated. In the second stage, the chi-square minimization estimation procedure proposed by Hamilton and Wu (2012) is applied to recover the coefficients of the related model (using the equivalences between them). Finally, we solve for the latent factors using Equation 21.

3 Data

To estimate the GATSM, we use the yields of zero-coupon public debt securities in nominal and real terms with a maturity of 1, 6, 12, and 36 months. These series are from January 2003 to September 2022 and we use the yields observed on the last day of each month. **Table 1** presents a summary of their basic statistics for the nominal bonds that are the one with greater liquidity between the two markets. The series show a positive slope of the yield curve, a standard deviation that increases with maturity, high levels of autocorrelation, mainly in bonds with less maturity, and a greater cross-correlation between bonds whose maturity is close. Finally, the series have heavier tails than a normal distribution and show a positive asymmetry that grows with maturity[18].

Table 1. Statistical summary of the nominal yield curve.

	Statistics				Serial correlation			Cross-sectional correlation			
	Mean	Stand. Deviation	Skewness	Kurtosis	Lag 1	Lag 2	Lag 3	1 month	6 months	12 months	36 months
1 month	5.381	2.000	0.520	2.755	0.970	0.941	0.907	1	0.971	0.907	0.714
6 months	5.813	2.058	0.615	2.249	0.976	0.951	0.921	0.971	1	0.981	0.853
12 months	6.271	2.181	0.737	2.123	0.975	0.944	0.914	0.907	0.981	1	0.937
36 months	7.551	2.580	0.956	2.798	0.967	0.927	0.889	0.714	0.853	0.937	1

Note: the left side of the table presents a statistical summary of the yields used in the GATSM estimation. The central and right sides show these yields' serial and cross-sectional correlation, respectively. The series are from January 31 of 2003 to September 30, 2022.
Source: Bolsa de Valores de Colombia (BVC), authors' calculations.

Considering that the GATSM works under the expectation hypothesis (EH) non-compliance, we carried out some empirical approximations to test this assumption for the nominal bond market. The results are shown in Appendix 1. According to the three exercises proposed by Campbell (2018), we found statistical evidence against the EH. On the other hand, since the estimation method that this work follows seeks to calculate the parameters of structural equations 23 and 24, and the identification process of the GATSM depends on these equations, verifying the stability of these coefficients is necessary. For this, a set of structural change tests were carried out (Andrews F statistic, 1993; Bai and Perron, 2003, CUSUM statistic and MOSUM statistic; Appendix 2) for the yield residuals that are estimated without error, following the specification of structural equations previously proposed. We verified the absence of structural change in these series.

4 Results

Considering the specification exposed in the methodological section, the yields at 1, 12, and 36 months were categorized as observed without error, and the yield at six months as observed with error[19]. Equations 17 (time-varying risk premium), 13 (unobserved dynamic factors), 16 (short-term TES bond yield), and 21 (TES bond price for any maturity *n*) are constructed with the coefficients obtained (Table 2).

[18] The results are the same for the UVR (real) yield curve. In particular, UVR yields exhibit more significant skewness and kurtosis, and smaller correlation levels than COP (nominal) yields.
[19] The choice of this specification followed two criteria: the recommendation made by Hamilton and Wu (2012), who take the exact specification, and the results of different in-sample and out-of-sample goodness-of-fit tests performed for 20 different specifications (Appendix 3).

Table 2. GATSM coefficients[20]

Panel A. nominal yield curve

Variable	**Coefficient** *p-value*		
c^Q	-0.0373***	0.166416**	0.31694***
	0.00	**0.03**	**0.00**
ρ^Q	0.995276***		
	0.00		
	-0.055764**	0.944387***	
	0.04	**0.00**	
	-0.033739***	0.146223***	0.891577***
	0.00	**0.00**	**0.00**
ρ	0.902654***	-0.116856***	0.035889***
	0.00	**0.00**	**0.00**
	-0.055581***	0.851818***	0.053788***
	0.00	**0.00**	**0.00**
	0.033235***	0.157486***	0.899004***
	0.00	**0.00**	**0.00**
δ_0	0.004281***		
	0.00		
δ_1	0.000022***	0.000067***	0.000378***
	0.00	**0.00**	**0.00**
λ	0.0373***	-0.166416***	-0.31694***
	0.00	**0.03**	**0.00**
Λ	-0.092622	0.000183***	0.066974
	0.17	**0.00**	**0.61**
	-0.116856***	-0.092568***	0.011262**
	0.00	**0.00**	**0.02**
	0.035889***	0.053788***	0.007426**
	0.00	**0.00**	**0.04**

Panel B. real yield curve[21]

Variable	**Coefficient** *p-value*		
c^Q	-0.036813	-0.138832**	0.346351***
	0.71	**0.01**	**0.00**
ρ^Q	0.993009***		
	0.00		
	0.058575	0.940607***	
	0.22	**0.00**	
	-0.008395	-0.161181**	0.867674***
	0.58	**0.01**	**0.00**
ρ	0.891417***	0.084176	0.042362
	0.00	**0.31**	**0.43**
	0.097488	0.826454***	-0.061085
	0.22	**0.00**	**0.29**
	-0.016491	-0.09877**	0.732266***
	0.94	**0.01**	**0.00**
δ_0	0.000832***		
	0.00		
δ_1	0.000061***	0.000006***	0.000521***
	0.00	**0.00**	**0.00**
λ	0.036813	0.138832**	-0.346351***
	0.71	**0.01**	**0.00**
Λ	-0.101591***	0.038913	-0.008096
	0.00	**0.28**	**0.92**
	0.084176	-0.114153***	0.062411
	0.31	**0.00**	**0.22**
	0.042362	-0.061085	-0.135409***
	0.43	**0.29**	**0.00**

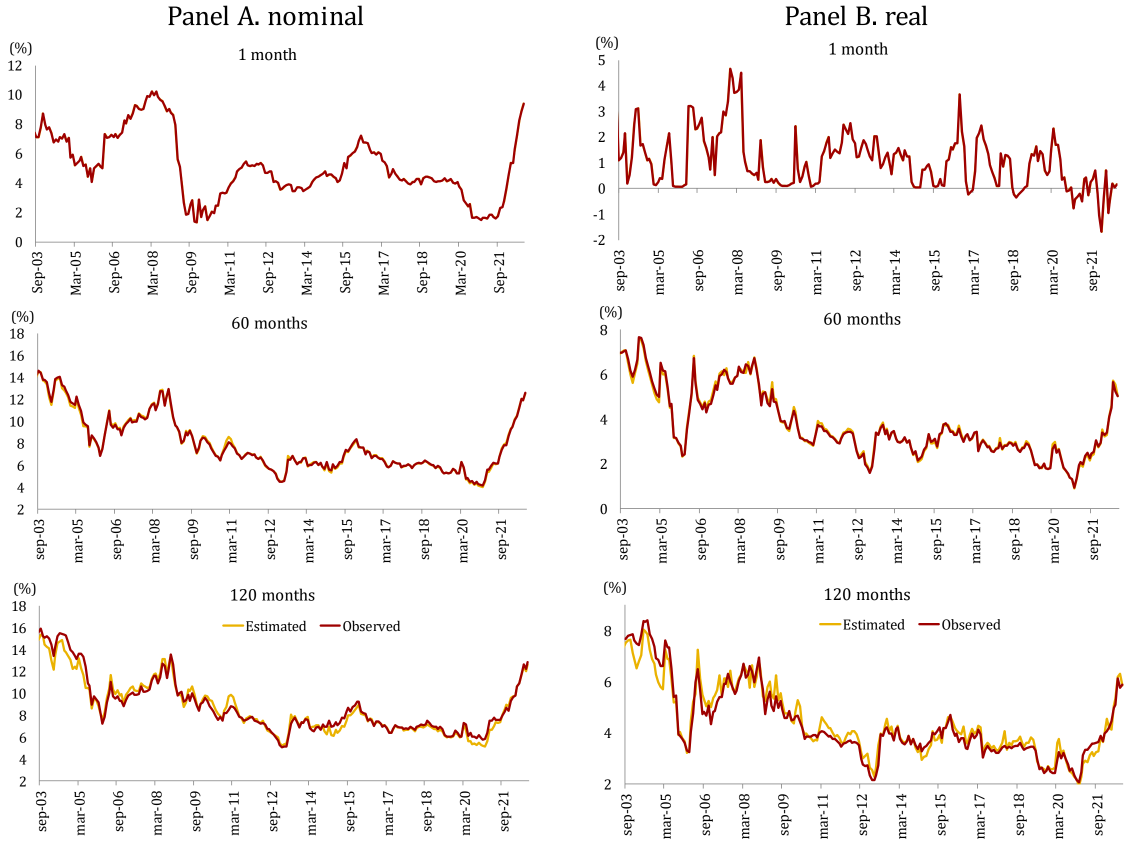
Note: these tables present the COP (nominal) and UVR (real) yield curve system of equations coefficients' (Equations 13, 16, and 17) and their significance p-value. The symbols *, **, *** denote statistical significance at the 10%, 5% and 1% level, respectively. Source: Authors' calculations.

The evolution of the estimated yields from the GATSM versus those observed in the Colombian public debt market for 1, 60, and 120 months is presented below. The model fit is adequate, and its performance increases in the short and medium term of the yield curve (Figure 1).

[20] We obtained confidence intervals following a bootstrapping procedure in the vein of the one exposed in Appendix 5.

[21] UVR denotes *Unidad de Valor Real* and refers to inflation-indexed bonds.

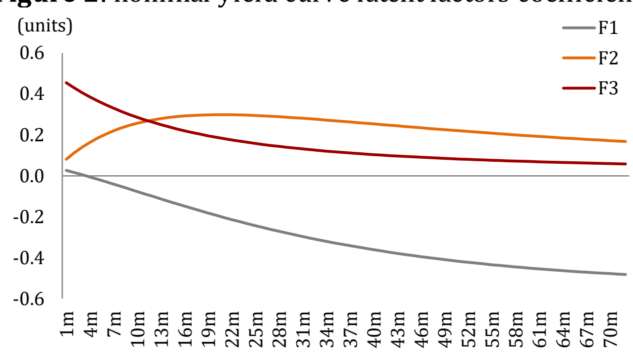
Figure 1. Fitted vs. observed yield curve.



Note: 1-,60-, and 120-months observed and estimated yields for the COP (nominal) and UVR (real) bonds are shown to evaluate the in-sample goodness-of-fit.
Source: Authors' calculations.

By construction, TES bond yields follow an affine process to three unobserved dynamic factors (Equation 21). The bond yield at a given maturity depends only on a combination of coefficients and the three estimated factors. Figure 2 presents the coefficients accompanying each factor across maturities to understand better its contribution to the yield curve structure.

Figure 2. nominal yield curve latent factors coefficients.



Given that latent factors do not have financial interpretation ex-ante, the coefficients accompanying these factors are plotted across maturities to associate them with the COP (nominal) yield curve's slope, level, and curvature.
Source: Authors' calculations.

The weight of the second factor (F2) is the most stable, and it is almost flat across maturities; it can therefore be associated with the level factor of the curve. The weight accompanying third factor (F3) decreases as maturity increases, thus having a greater incidence on the curve in the short term; this interpretation is associated with a curve slope factor. Finally, the first factor (F1) differentially affects the short-term from the rest of the yield curve in absolute terms and signs; this changing effect is associated with a curvature factor of the zero-coupon curve of TES bonds.

4.1 Yield curve forecasting using macroeconomic variables

Satellite models have become an increasingly relevant tool to establish a link between different bank indicators and macroeconomic variables given a specific scenario, to then project the evolution of credit, liquidity, and market risk, among other variables. The Central Bank of Colombia employs several satellite risk models within its Systemic Stress Model, such as the credit risk, the interest rate risk, and the market risk satellite models (Gamba et al. 2017). To date, the latter focuses solely on the potential effects of a massive sale of securities held by a group of investors and leaves behind the movements of the yield curve in response to macroeconomic conditions. In this regard, the aim of this application exercise is to contribute to the market risk satellite model by developing one to forecast the TES yield curve given some macroeconomic variables. We propose a two-step exercise in which, in the first stage, we forecast the TES bond yields for a reduced group of maturities given a set of macroeconomic variables. In the second stage, we employ the GATSM developed above to build the whole yield curve given the yields provided in the previous step.

To forecast the TES bond yields, we follow Gross and Poblacion (2019) and Sala-I-Martin et al. (2004). Both papers employ the Bayesian Averaging of Classical Estimates (BACE), the first to relate credit risk measures to macroeconomic conditions and the second to analyze the determinants of long-term growth. The BACE approach is like the Bayesian model averaging (BMA) approach to the extent that it relies on the assumption that no single model is the true one. Thus, it combines into a posterior model the individual specifications through weights.

The individual models follow an Autoregressive Distributed Lag (ADL) structure in which the dependent variable Y_t corresponds to the TES bond yields and is a function of its own lags and a set of contemporaneous and lagged macroeconomic variables. This set includes the SYSMO model output variables[22]: GDP, monetary policy rate, consumer price index, inflation rate, inflation expectations, and five years Credit Default Swaps (CDS). The period of analysis comprises 2003m1 – 2022m12.

$$Y_t = \alpha + \rho_1 Y_{t-1} + \cdots + \rho_p Y_{t-p} + \sum_{k=1}^{k_i} \left(\beta_0^k X_t^k + \cdots + \beta_{q^k}^k X_{t-q^k}^k \right) + \epsilon_t \quad (34)$$

We estimate Equation 34 for 1-month, 12-month, 36-month, and 72-month yields. This is done considering that in the second stage we employ the GATSM to build the entire yield curve. Therefore, using the GATSM estimates along with a small set of yields, which are assumed to be represented by the model without error, we find the set of latent factors, and then the entire yield curve. For all individual equations i the lag structure (i.e. p and q) is chosen by estimating all possible combinations up to a limit L (we set it to one)[23], and subsequently selecting the model with the minimum Bayesian Information Criteria (BIC).

[22] The macroeconomic series are built using either an extended DSGE model following Smets and Wouters (2007) and Caldara et al. (2014) or a semi-structural model for monetary policy analysis and macroeconomic forecasting based on a New-Keynesian rational expectation framework for an oil-exporting small open economy presented in Guarín et al. (2020).

[23] Following Gross and Poblacion (2019), we conducted robustness checks by setting $L = 2$ and $L = 3$. Results remain fully robust mainly because the individual equations in the model tend to have no more than one significant lag of the macroeconomic variables.

The BACE approach requires setting an ex-ante maximum model dimension K, since each individual equation is estimated considering all possible combinations of the available macroeconomic variables X. We set $K = 2$, thus $I = \sum_{k=1}^{K} \frac{X!}{k!(X-k)!}$ models were computed. Once each individual equation i is estimated, they are combined in a single posterior estimation that is constructed as the weighted average of the individual estimates. Just as Gross and Poblacion (2019), the implemented weights were the BIC rather than a predictive performance measure.

$$E[\beta|y] = \sum_{i=1}^{I} P(M_i|y)\widehat{\beta_\iota} \tag{35}$$

Besides the posterior estimates, we construct the posterior variance to analyze the statistical significance of the variables included in the model.

$$Var[\beta|y] = \sum_{i=1}^{I} P(M_i|y)Var[\beta|y, M_i] + \sum_{i=1}^{I} P(M_i|y)\left(\widehat{\beta_\iota} - E[\beta|y]\right)^2 \tag{36}$$

In general, the relevant variables for the 12-month yield are the monetary policy rate and inflation expectations, while the CDS, inflation rate, and inflation expectations are suitable for the two remaining yields. For all three, the GDP estimate has a negative impact but is not statistically significant. Lastly, the estimation of the 1-month yield includes the inflation rate, inflation expectations and monetary policy rate (Appendix 4). These findings are aligned with several studies, since for the yields with shorter maturities variables related to the interest rate and inflation are adequate (Poghosyan 2014), while for the yields with longer maturities, the 5-year CDS gains relevance.

Once the posterior models are built, we proceed to forecast the yields 3-month and 6-month ahead[24] to thereafter construct the entire yield curve. For the latter, we employ the Equation 37 for different specifications (S).

$$\widehat{y_{t+\iota}^{n}} = -\overline{A_n^S} + \overline{B_n^S} F_{t+i}^{S} \tag{37}$$

$\overline{A_n^S}$ and $\overline{B_n^S}$ corresponds to the matrices A and B of Equation 21 estimated using the specification S for the yield n, and F_{t+i}^{s} corresponds to the factors associated with the specification S. For each different specification analyzed, we construct $\overline{A_n^S}$ and $\overline{B_n^S}$ using the coefficients presented in Table 2, and for $\widehat{y_{t+\iota}^{n}}$ we employ the yields predicted in the BACE model. From this, we calculate the factors associated with the yield curve structure as shown in Equation 38.

$$\overline{B_n^S}^{-1} * \left(\widehat{y_{t+\iota}^{n}} + \overline{A_n^S}\right) = \widehat{F_{t+\iota}^{S}} \tag{38}$$

Once the factors are found, we use once again Equation 37 to construct the entire yield curve for a 3-month and 6-month horizon. The output corresponds to a unique yield curve for each specification defined.

To test the performance of these results, we compare our findings to those obtained from the GATSM without including macroeconomic variables (i.e. without the BACE approach). For this, we use the Equation 13, in which the latent factors are assumed to follow a vector autoregressive model (VAR), to forecast the latent factors associated to each S specification 3 months and 6 months ahead. Thereafter, we include those forecasted factors in Equation 37 to build the entire yield curve. As a result, we get different yield curves for each specification given.

[24] These horizons were selected since the projection period of the SYSMO model is quarterly.

To compare the accuracy of both methodologies, Table 3 depicts several forecast error measures comparing the observed yield curve 3 months and 6 months ahead starting from December 2022 with that predicted following the two-step BACE+GATSM approach and solely the GATSM approach. Considering that the in-sample and out-of-sample goodness-of-fit tests find the specifications {1,12,36}, {1,12,72}, and {1,12,36} suitable[25], we perform the exercise for these maturities. Results are as follows: in general, the BACE+GATSM approach shows the lowest forecast error measures in both projection periods for the 1-month, 12-month, and 72-month triplet of yields. Moreover, for the 6-month horizon, it reflects a better performance than the GATSM regardless of the specification.

Table 3. Forecast error measures

Model	Yields	3-month forecast				6-month forecast			
		MAFE	MAFRE	RMSFE	RMSFPE	MAFE	MAFRE	RMSFE	RMSFPE
GATSM	1,12,36	1.0340	0.0940	1.1020	0.1000	2.4298	0.2449	2.4755	0.2490
BACE	1,12,36	1.3953	0.1251	1.4885	0.1318	**0.6773**	**0.0682**	**0.7136**	**0.0717**
GATSM	1,12,72	1.0530	0.0957	1.1294	0.1024	2.4574	0.2477	2.5060	0.2521
BACE	1,12,72	**1.0464**	**0.0950**	**1.0685**	**0.0964**	**0.6319**	**0.0637**	**0.6635**	**0.0668**
GATSM	1,36,72	0.9887	0.0894	1.0797	0.0973	2.3921	0.2409	2.4577	0.2470
BACE	1,36,72	1.1849	0.1084	1.2317	0.1127	**0.6327**	**0.0638**	**0.6663**	**0.0670**

The table depicts different forecast error measures for the yield curve estimated using the BACE approach and the GATSM approach. The aim is to compare the observed yield curve 3 months and 6 months ahead, starting from December 2022, with that predicted according to each proposed approach. Authors' calculations.

These findings suggest that our proposed methodology is a useful application tool to predict the yield curve given some macroeconomic conditions. Compared to the GATSM, it improves the forecast accuracy. In this sense, we intend to contribute to the SYSMO market risk satellite model since the proposed model can forecast the yield curve movements in response to different macroeconomic scenarios. In the case of the SYSMO, the resulting yield curve would be consistent with a 'tail-risk' scenario in which a range of severe adverse shocks will materialize (e.g. increasing interest rates, high inflation rate, low GDP growth, among others).

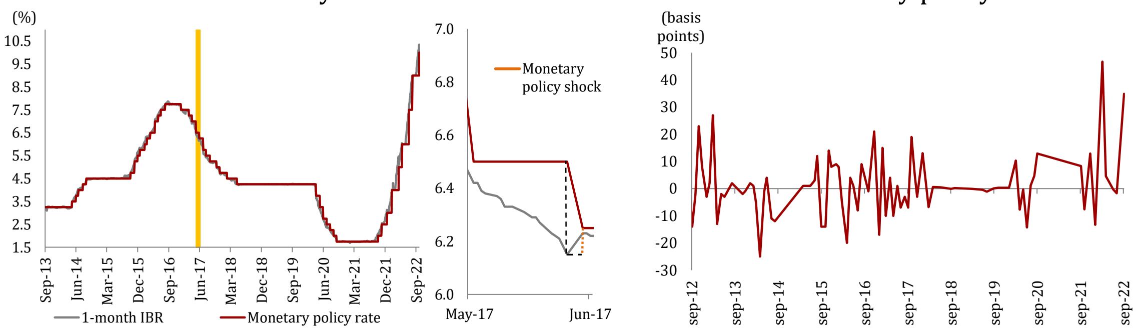
4.2 Unexpected monetary policy shocks on the yield curve.

Unlike the approaches of Ang and Piazzesi (2003), Diebold, Piazzesi and Rudebusch (2005), Rudebush and Wu (2008), who use the arbitrage-constraint-free Taylor Rule derived from their yield curve model to identify monetary policy shocks, here an unexpected shock is identified as the difference between the one-month banking lending rate (*IBR* in Spanish) observed on the day the Board of Directors of the Central Bank of Colombia (JDBR) decides on the policy rate and the new value of the last (Figure 3).

[25] See Appendix 3.

Figure 3. IBR and monetary policy rate.

Panel A. Dynamics Panel B. Monetary policy shocks[26]



The monetary policy shock is the difference between the 1-month banking lending rate (IBR) observed on the day of the decision and the monetary policy rate Banco de la República decides during the meeting.

Source: Banco de la República, authors' calculations.

This identification process follows the approach of Cochrane and Piazzesi (2002) for obtaining policy shocks and the conceptual framework of Nakamura and Steinson (2018). In this way, it aims to reduce the endogeneity generated by the omitted variable problem in the Taylor Rule derived from the affine model in an analogous way to any VAR model (Nakamura and Steinson, 2018). This strategy of identifying the unexpected monetary shock assumes that market agents and the JDBR have access to a similar set of information about the current state of the economy and its future dynamics and, therefore, any difference between the one-month interbank lending rate observed on the day of the monetary policy decision and the new monetary policy rate corresponds to a monetary policy surprise for the market[27].

After identifying the shock, the second step is to choose the specification to find the impulse-response functions. In this paper, we take the Jordà (2005) approach as it is less restrictive in the structure of momentum-generating functions (Nakamura and Steinson, 2018). Cardozo *et al*. (2020) use it to quantify whether the increasing participation of foreigners in the local government bond market after 2014 affected the transmission of monetary policy for these securities. In particular, the authors analyze the impact on the 10-year sovereign debt bond, while the approximation of this paper using the estimated GATSM allows us to find the effect on the whole nominal and real yield curve.

The procedure is as follows:

1. The effect of the monetary policy surprises on the dynamic factors of the GATSM is estimated following the approach of Jordà (2005):

$$F_{t+h} - F_t = \alpha_h + \beta_h \epsilon_t + \mu_{t+h}$$

   Where h denotes the number of days after the policy surprise, and ϵ_t is the magnitude of the surprise at time t.

2. The effect is mapped to the yield curve using the GATSM structure (Equation 21):

$$IRF_h = Y_\epsilon - Y$$
$$IRF_h = A_n + B_n F_\epsilon - (A_n + B_n F)$$
$$IRF_h = B_n (F_\epsilon - F)$$
$$IRF_h = B_n \beta_h \epsilon_t$$

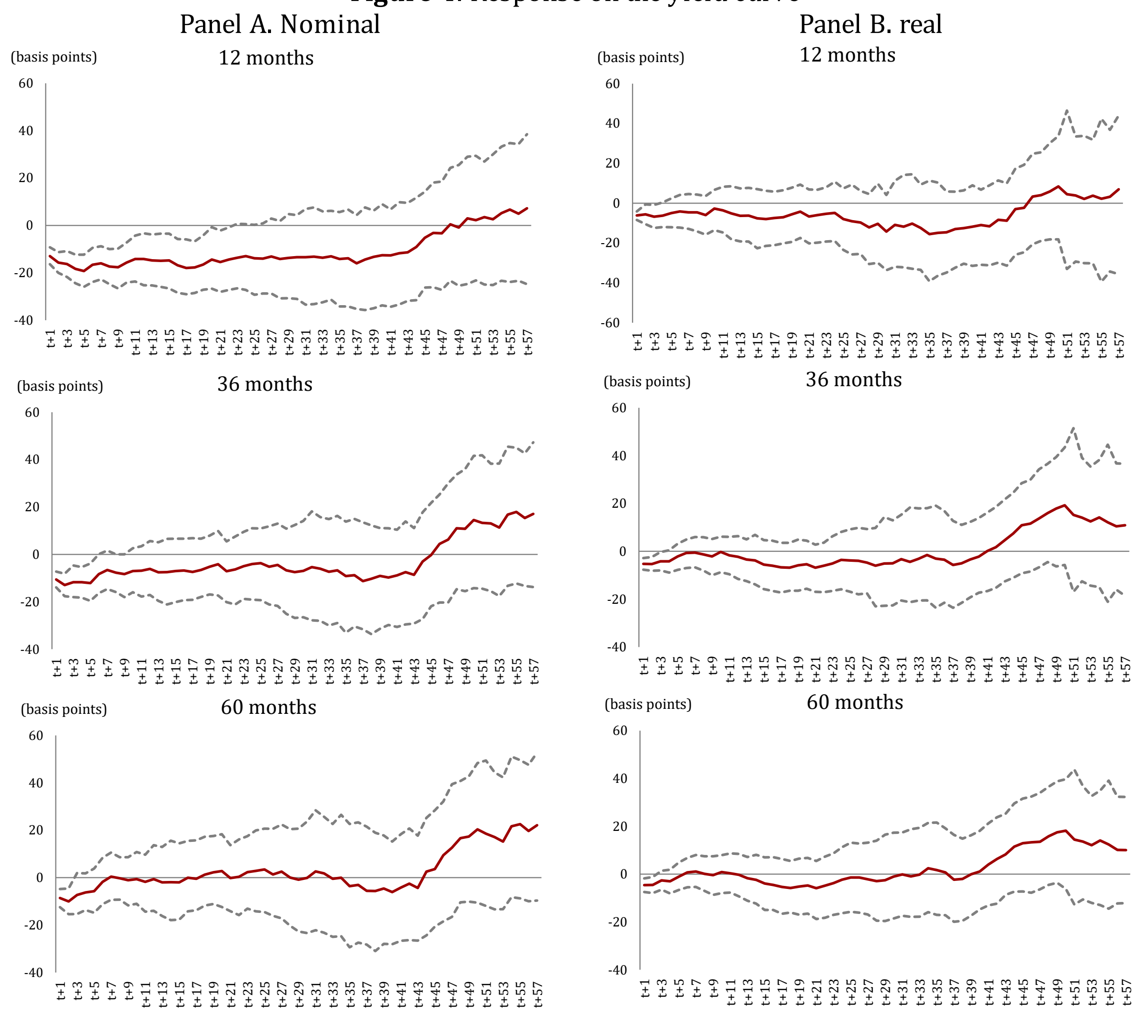
[26] For this exercise, only the days when there was a policy decision, and the value of the shock was different from zero were taken. Thus, we obtained a series that starts on August 24, 2012, and includes 81 surprises, with the last on September 29, 2022.

[27] The advantage of this alternative is that in addition to correcting the omitted variable problem, it also considers that the weight assigned to each variable that the JDBR observes can change over time.

Where Y_ϵ is the yield with maturity *n after the shock,* F_ϵ is the set of shocked dynamic factors, B_n is the matrix of coefficients accompanying the factors in Equation 21, and β_h is the coefficient estimated in step 1, reflecting the impact of a policy surprise on the variance of the dynamic factors.

This exercise assumes an unexpected shock of 20 basis points, i.e., expansionary and in the range of observed shocks. The 1-, 3- and 5-year yield response in nominal and real terms is presented in Figure 4. The analysis horizon is up to 90 days after the policy surprise, and the confidence intervals obtained by bootstrapping are reported (Appendix 5). For the nominal yield curve, we found that the shock is transmitted almost entirely over the curve and immediately, being this response statistically significant during the first 20 days in the case of the 12-month yield, five days in the case of the 36-month yield, and two days in the case of the 60-month yield. For the real yield curve, the response of the curve is usually between 4 and 6 basis points, with statistical significance during the first three days after the shock.[28]

Figure 4. Response on the yield curve

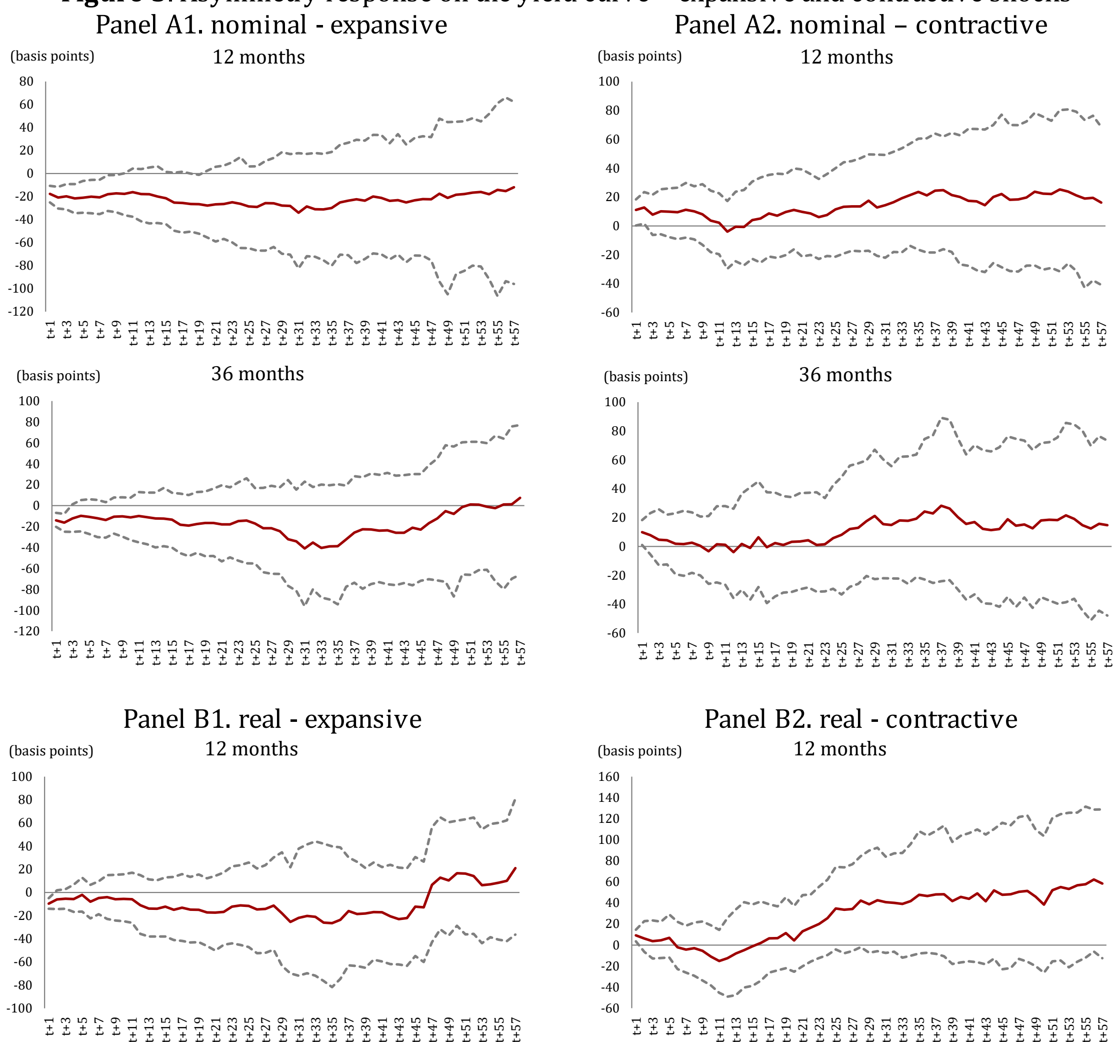


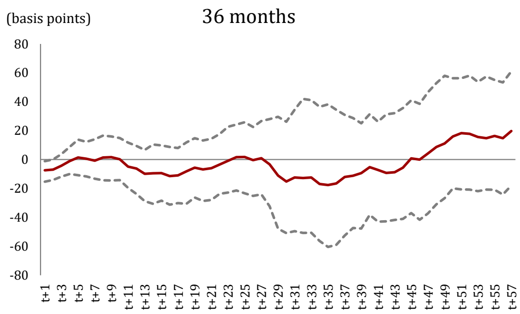
Given a 20 basis points expansive unexpected monetary policy surprise, the graph depicts the impulse-response functions (IRF) on 12-, 36-, and 60-months COP (nominal) and UVR (real) yield curve. Confidence intervals were generated by bootstrapping.
Source: Authors' calculations.

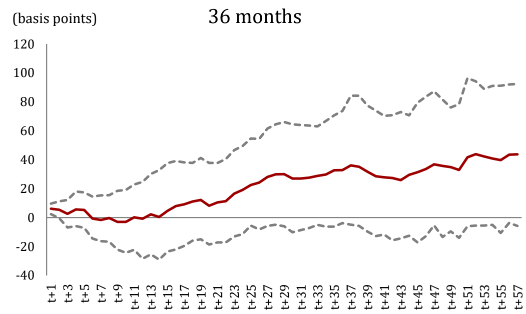
[28] The recent period has been characterized by the Covid-19 health crisis and the most significant policy surprises, according to Figure 3. For comparison, we estimated the GATSM with information up to February 2020. The comparison suggests that the phenomena of recent years have not influenced the magnitude and speed at which a policy surprise is transmitted on the yield curve. However, the time horizon over which this impact is significant has shortened.

Finally, we repeat the previous impulse-response exercise differentiating periods of expansionary monetary policy shocks (the new monetary policy rate is lower than the observed IBR rate) from contractionary shocks, to evaluate a possible asymmetry in monetary policy transmission on the yield curve. Compared to the aggregate exercise, the assessment of contractionary shocks is made assuming an unexpected negative surprise of 20bp (Figure 5). For the expansionary policy surprises, we found a statistical significance of the response up to two days for the real yield curve and up to 20 days for the nominal yield curve. The magnitude of the transmission is complete for the nominal yield curve. As for the contractionary surprises, the significance is also two days for the nominal and real yield curves, and in no case does the magnitude of the response approach the volume of the shock. Consequently, we found an asymmetry in monetary policy transmission, where expansionary shocks tend to have a greater and faster impact on the public debt yield curve.

Figure 5. Asymmetry response on the yield curve – expansive and contractive shocks







Expansive and contractive monetary policy shocks are evaluated separately to assess an asymmetry in monetary policy transmission on the TES bond yield curve. Confidence intervals were generated by bootstrapping.
Source: Authors' calculations.

5 Concluding remarks

GATSM is a useful tool not only to model the public debt yield curve based on two or three latent factors but also to study the impact of certain macroeconomic and financial conditions on the yield curve. In this regard, the GATSM has been used to measure interest rate spreads, analyze the effects of macroeconomic shocks, depict the monetary policy rule, examine behavior patterns in long-term yields, derive inflation expectations, evaluate the effectiveness of central bank interventions, among other applications.

We contribute to the existing literature on asset pricing by offering an updated parsimonious estimation of the GATSM, following Hamilton & Wu (2012), with potential applications to improve the understanding of the relationship between the yield curve and microfinance dynamics. Regarding the applications, the two practical exercises contribute, the first to improve the mapping of the yield curve given a scenario, and the second to analyze the impact of monetary policy surprises on the yield curve.

Further application exercises can be performed, such as analyzing the impact of unexpected inflation shocks, economic growth, or relevant economic news on the yield curve. Likewise, a high-frequency model capable of generating estimates of the intraday yield curve could be developed allowing the evaluation of unexpected information in real-time. In addition, the dynamics and forecast of the exchange rate between two countries could be studied using the non-arbitrage restrictions and developing additional identification conditions over a joint model estimation for the two countries (i.e., Colombia and the United States).

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Appendix

Appendix 1. Expectation hypothesis tests

This appendix illustrates the statistical exercises that concluded the non-fulfillment of the expectation hypothesis (EH) for the Colombian public debt yield curve. These are based on what is proposed by Campbell (2018) in Chapter 8.2 *The Expectations Hypothesis of the Term Structure*. We define the yield spread for a maturity n (Equation A.1), the holding period return (hpr) for one year (Equation A.2 and A.3), and the spread over the *hpr* (Equation A.4) as:

$$s_{n,t} = y_{n,t} - y_{1,t} \tag{A.1}$$

$$\begin{aligned} r_{n,t+1} &= p_{n-1,t+1} - p_{n,t} \\ &= ny_{n,t} - (n-1)y_{n-1,t+1} \\ &= y_{n,t} - (n-1)(y_{n-1,t+1} - y_{n,t}) \end{aligned} \tag{A.2}$$

Adding and subtracting $(n-1)y_{n,t+1}$:

$$\begin{aligned} r_{n,t+1} &= y_{n,t} - (n-1)(y_{n-1,t+1} - y_{n,t}) - (n-1)y_{n,t+1} + (n-1)y_{n,t+1} \\ &= y_{n,t} - (n-1)(y_{n,t+1} - y_{n,t}) + (n-1)(y_{n,t+1} - y_{n-1,t+1}) \end{aligned} \tag{A.3}$$

$$r_{n,t+1} - y_{1,t} = s_{n,t} - (n-1)(y_{n-1,t+1} - y_{n,t}) \tag{A.4}$$

Since this hypothesis states that the expected excess return of long-term bonds over short-term bonds is constant over time, the log-EH can be written for a holding period in period n:

$$E_t\left[y_{nt} - \left(\frac{1}{n}\right)\sum_{i=0}^{n-1} y_{1,t+i}\right] = \theta_n \qquad \text{(A.5)}$$

The first term on the left is the return on a bond with maturity n that is held to maturity, while the second term is the return on a single-period bond that is reinvested for n periods. According to the *hypothesis*, the difference between the two terms is a constant θ_n that depends only on maturity n. Now, the implications of the hypothesis can be used to test its existence. Following Campbell (2018), this is done in three ways. First, the EH implies that:

$$s_{nt} = (n-1)E_t\left[y_{n-1,t+1} - y_{nt}\right]$$

The equation states that the yield spread between a bond with maturity n and a single-period bond is represented by $s_{nt} = y_{nt} - y_{1t}$. It means that the current difference between the yields of two bonds with varying maturities (n and 1) should equal the bond's return with the higher maturity (n) from today to tomorrow, multiplied by the remaining periods of its maturity. This statement can be verified through the following equation:

$$(n-1)\left(y_{n-1,t+1} - y_{nt}\right) = \alpha + \beta * s_{nt} + u_{t+1}$$

The hypothesis that β=1 would confirm the first implication of the EH about changes in long-term performance. The second exercise involves rearranging equation A.5 to reach the following conclusion:

$$s_{nt} = E_t\left[\left(\frac{1}{n}\right)\sum_{i=0}^{n-1}\left(y_{1,t+i} - y_{1t}\right)\right] = E_t\left[\sum_{i=0}^{n-1}\left(1 - \frac{1}{n}\right)\Delta y_{1,t+i}\right]$$

Hence, when the s_{nt} (yield spread) is high, the average short-term interest rate over the life of the long-term bond should be substantially higher than the short-term interest rate. This can be tested with the following equation:

$$\sum_{i=0}^{n-1}\left(1-\frac{1}{n}\right)\Delta \mathrm{y}_{1,\mathrm{t+i}} = \alpha + \beta * s_{nt} + u_{t+n}$$

Again, the hypothesis β=1 would confirm the second implication of the EH regarding changes in the short-term rate. The third exercise involves the forward rate, defined as $f_{n,t} = (n+1) * y_{n+1,t} - n * y_{n,t}$. Combining this equation with equation A.5, we obtain:

$$f_{n,t} = \theta_n + E_t[y_{1,t+n}]$$

This suggests that the logarithm of the forward rate is equivalent to a constant (which depends only on the maturity) and the expectation of the future rate of the short-term bond, so the forward rate can be used to understand the market's expectations of future short-term rates, net of the risk premium. Subtracting the short-term rate from both sides of the above equation, we get:

$$y_{1,t+n} - y_{1,t} = \alpha + \beta * (f_{n,t} - y_{1,t}) + \varepsilon_{t+n}$$

In this way, Campbell proposes to test again if β=1, in which case the expectations hypothesis would be fulfilled. Finally, the three equations are estimated by Ordinary Least Squares, and due to the overlap present in the errors, these are corrected by the Newey-West estimator. Table A1.1 presents the results of these three tests. The first row shows the estimated coefficient, while the second presents the robust standard error. Evidence against the EH compliance is found due to coefficients are not statistically equal to one.

Table A1.1. Summary of expectations hypothesis tests.

Equation	Maturity (years)				
	1	3	5	7	10
Long-yield	-0.239	-0.537	-1.096	-1.610	-2.238
changes	(0.665)	(0.787)	(1.122)	(1.524)	(2.334)
Short-yield	-0.281	-0.013	-0.382	-0.130	-2.704
changes	(0.92)	(0.643)	(0.351)	(0.452)	(0.628)
Forward rate as	0.444	0.101	0.244	0.852	0.712
future yield predictor	(0.202)	(0.299)	(0.101)	(0.059)	(0.004)

Note: the expectation hypothesis is tested in three ways (equations), and results are shown for five maturities. EH is rejected if the coefficient is not statistically significant to one. The standard deviations are in parenthesis and were estimated with the Newey-West estimator.
Source: Authors' calculations.

Appendix 2. Structural change tests on model errors

The estimated structural equations are presented below:

$$
\begin{aligned}
Y_t^{1\,month} &= \bar{A}^*_{1\,month} + \phi^*_{1\,month,1} Y_{t-1}^{12,36\,months} + \epsilon^*_{1\,month,t} \\
Y_t^{12\,months} &= \bar{A}^*_{12\,month} + \phi^*_{12\,month,1} Y_{t-1}^{1,36\,months} + \epsilon^*_{12\,months,t} \\
Y_t^{36\,months} &= \bar{A}^*_{36\,months} + \phi^*_{36\,months,1} Y_{t-1}^{1,12\,months} + \epsilon^*_{36\,months,t}
\end{aligned}
$$

In the case of Andrews (1993), the aim is to generalize Chow's test for structural change. The objective is to test whether it is a breakpoint in the following model:

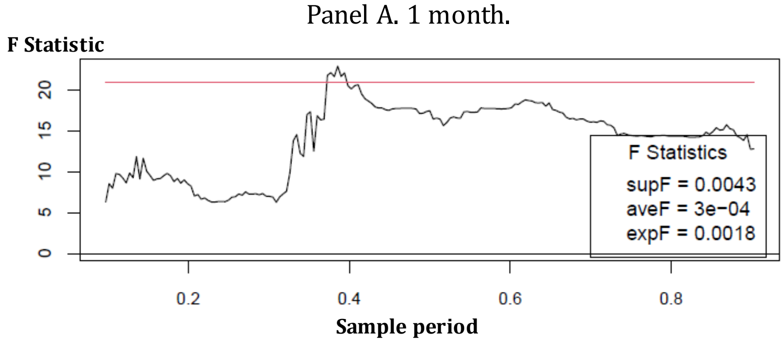
$$
\begin{gathered}
Y_t = \beta X_t + u_t \\
t \in 1, \dots, T \\
\beta = [\beta_1, \dots, \beta_K]
\end{gathered}
$$

A primary starting point is to evaluate the F statistic:

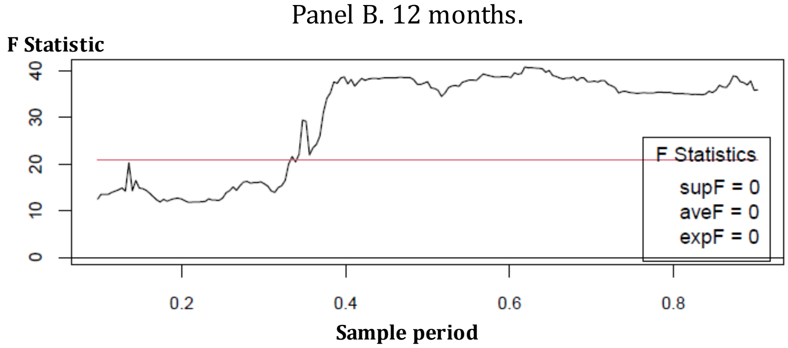
$$F_i = \frac{u^T u - e^T e}{e^T e/(T-2k)}$$

The test identifies a point as the potential structural change when the statistic reaches its maximum value. The null hypothesis states that there is no structural change. The results are shown below and suggest evidence of structural change for yields at all three maturities:

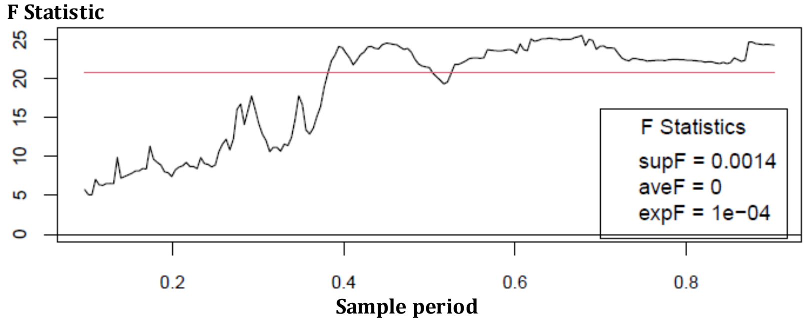
Figure A2.1. Andrews' structural change test
Panel A. 1 month.



Panel B. 12 months.



Panel C. 36 months.

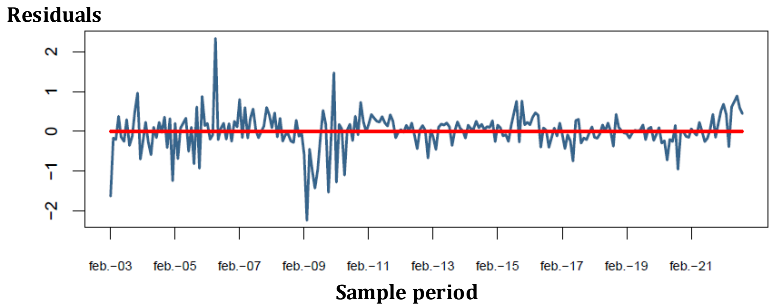


The sample period is rescaled to 0-1 unit, and the test is estimated at each point. A structural change can be concluded if the F Statistic overcomes the threshold (red line).
Source: Authors' calculations.

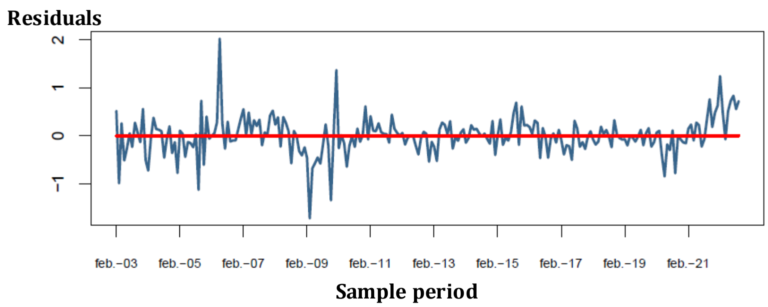
As for the Bai and Perron test (2003), the results suggest the absence of structural change on the residuals of the estimated model (a structural change would be denoted by a break in level or slope above the red line):

Figure A2.2. Bai & Perron structural change test

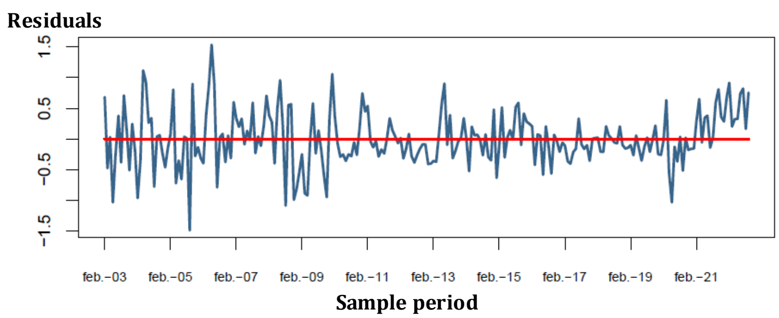
Panel A. 1 month



Panel B. 12 months



Panel C. 36 months.



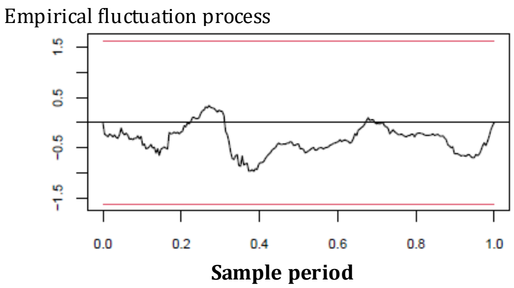
Bai & Perron test for structural change is carried out to 1-,12- and 36-month GATSM yields residuals. If the red line exhibits breaks, a structural change can be concluded.
Source: Authors' calculations.

Two additional tests are performed to identify structural changes. First, through the generalized fluctuations methodology CUSUM (*Cumulative Sum*). This methodology compares the mean of different subsamples with the mean of the total sample. Therefore, when the statistic exceeds the thresholds, it is inferred that the mean of at least one subsample is significantly different from that of the rest of the sample, evidencing a structural change at some point in the series. Second, MOSUM processes (which include a moving average for error estimation) are better at identifying structural changes in the estimated series because, unlike CUSUM processes, where the cumulative sums of residuals are less sensitive to changes in the parameters as the length of the series increases, all moving sums are equally sensitive to changes in the parameters.

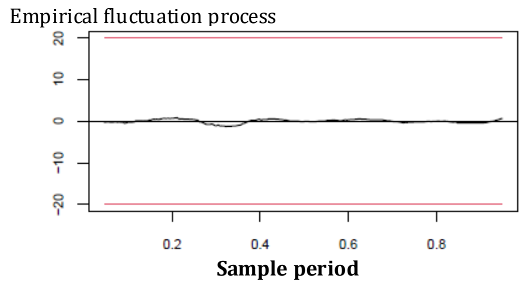
Figure A2.3. CUSUM and MOSUM structural change tests

Panel A. 1 month.

CUSUM test

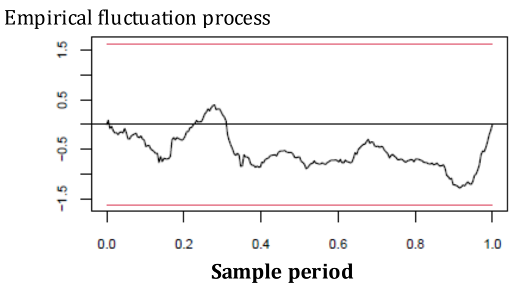


MOSUM test

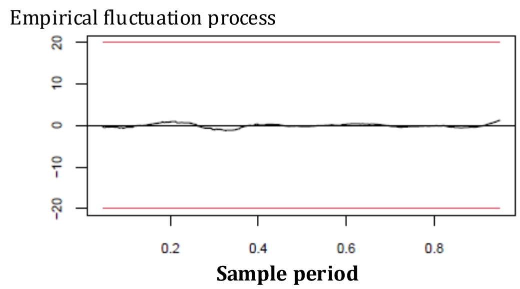


Panel B. 12 months.

CUSUM test

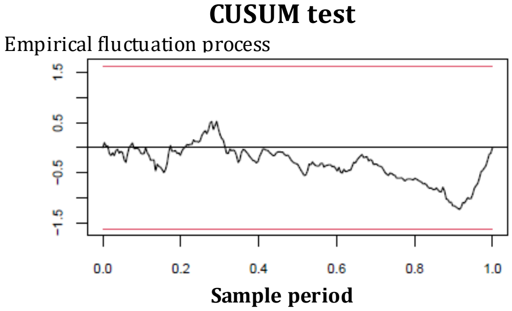


MOSUM test

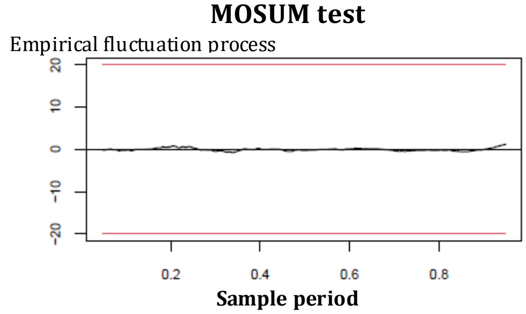


Panel C. 36 months.

CUSUM test



MOSUM test



MOSUM and CUSUM structural change tests are carried out for 1-, 12- and 36-months GATSM yields residuals. A structural change can be concluded if the black line goes out of the intervals (red lines).

Source: Authors' calculations.

A break or structural change in the series is identified when the statistic is located outside the intervals demarcated by the red lines. We found no evidence of structural change for the series analyzed. Therefore, the document works with the entire sample considered, that is, between January 2003 and September 2022.

Appendix 3. In-sample and Out-of-sample goodness-of-fit exercises for the selection of the GATSM specification

This section describes the exercises performed on the GATSM estimated using different specifications to evaluate their goodness-of-fit for the observed returns. The obtained results are the base for the GATSM specification selection in the empirical applications carried out in this work. These exercises are grouped into in-sample and out-of-sample assessments but have in common that in both cases, four measures of forecast error are calculated:

$$\boldsymbol{MAFE_m} = \frac{\sum_{i=1}^{n} \left| Y_{i,m}^{obs} - Y_{i,m}^{est} \right|}{n} \qquad \boldsymbol{MAFRE_m} = \frac{\sum_{i=1}^{n} \left| \frac{Y_{i.m}^{obs} - Y_{i,m}^{est}}{Y_{i,m}^{obs}} \right|}{n}$$

$$\boldsymbol{RMSFE_m} = \sqrt[2]{\frac{\sum_{i=1}^{n} \left(Y_{i,m}^{obs} - Y_{i,m}^{est} \right)^2}{n}} \qquad \boldsymbol{RMSFPE_m} = \sqrt[2]{\frac{\sum_{i=1}^{n} \frac{\left(Y_{i,m}^{obs} - Y_{i,m}^{est} \right)^2}{Y_{i,m}^{obs}}}{n}}$$

where $Y_{i,m}^{obs}$ is the observed return i at maturity m, $Y_{i,m}^{est}$ is the estimated return i at maturity m, m are the maturities at which goodness-of-fit is evaluated, which for this case are 1, 6, 12, 36, 60, 72, 120 y 240 months, y $i = 1, \dots, n$ are points in the time horizon at which the exercise is performed.

For the in-sample exercise, the entire analysis period was taken ($n = 237$), the GATSM was estimated in 20 different specifications, and each measure was calculated for each maturity m of analysis. Finally, the aggregate criteria were obtained by averaging the values of the eight maturities considered.

For the out-of-sample exercise, the following algorithm was applied for each of the 20 specifications considered:

1. The base is cut off 24 months before the analysis cut-off date, i.e., from January 2003 to October 2020.

2. The GATSM is estimated, and the unobserved dynamic factors $(\boldsymbol{F_t})$ are derived from the coefficients obtained.

3. Equation 13 forecasts the factors three months ahead, i.e., between November 2020 and January 2021.

4. These forecasts entered Equation 22 to generate estimated out-of-sample returns.

5. The four forecast error measures are calculated to compare the predicted yields with those observed.

6. The following month is added to the sample, and steps 1-5 are repeated until the 24 months initially extracted are covered. In this way, the algorithm is run 23 times for each specification (because in month 24 no forecast is made).

7. Each aggregate measure is obtained as the average of the values obtained individually across maturities and forecast horizons.

The results of these two exercises are shown in Table A3.1. We conclude that specification $Y_1 = \{1,12,36\}$ was appropriate as the baseline GATSM from the in-sample and out-of-sample forecast errors. Moreover, specifications with three yields are preferred for applications where the bond yield curve is modeled under a macroeconomic scenario because they allow linking macroeconomic variables with points on the curve with greater precision. In this sense, the Y_1 specifications {1,12,36}, {1,12,72}, and {1,36,72} reflect adequate forecast errors. Consequently, the three options were considered as alternatives for this exercise. Finally, we chose the specification for the monetary policy shocks application that focused on modeling the short-term yield curve with a maximum of maturities. This led us to take the baseline GATSM $Y_1 = \{1,12,36\}$ for impulse-response estimations.

Table A3.1. Forecast errors summary

	In-sample				Out-sample			
Maturities Y1	**MAFE**	**MAFRE**	**RMSFE**	**RMSFPE**	**MAFE**	**MAFRE**	**RMSFE**	**RMSFPE**
1, 12	4.5	0.5	2.7	0.3	4.0	0.7	1.7	0.3
1, 36	2.5	0.3	1.9	0.2	4.4	0.7	1.9	0.3
1, 72	1.6	0.2	1.3	0.1	>100	61.5	>100	57.5
1, 120	1.9	0.3	1.2	0.1	>100	19.9	57.1	12.4
12, 36	2.5	0.3	2.3	0.2	5.1	0.7	2.2	0.3
12, 72	1.8	0.2	1.5	0.2	5.4	0.8	2.3	0.3
12, 120	2.0	0.3	1.4	0.2	13.7	5.4	10.0	4.8
36, 72	3.0	0.5	2.1	0.4	12.3	4.0	7.0	2.9
36, 120	3.3	0.6	2.3	0.5	>100	>100	>100	>100
72, 120	4.8	0.9	3.2	0.9	>100	41.4	95.8	32.9
1, 12, 36	1.6	0.2	1.4	0.1	5.3	0.8	2.3	0.3
1, 12, 72	1.3	0.1	1.3	0.1	5.6	0.8	2.6	0.4
1, 12, 120	1.3	0.2	1.0	0.1	>100	91.8	>100	87.9
1, 36, 72	1.3	0.2	1.2	0.1	7.0	1.2	3.1	0.6
1, 36, 120	1.1	0.1	0.9	0.1	>100	>100	>100	>100
1, 72, 120	73.6	14.6	53.5	12.0	>100	>100	>100	>100
12, 36, 72	>100	>100	>100	>100	>100	>100	>100	>100
12, 36, 120	>100	44.3	>100	48.3	>100	>100	>100	>100
12, 72, 120	>100	>100	>100	>100	>100	>100	>100	>100
36, 72, 120	2.6	0.4	1.8	0.3	>100	>100	>100	>100

Twenty GATSM specifications are considered and estimated to determine, by in-sample and out-sample goodness-of-fit, the best baseline model and the best alternatives for the two empirical applications this work performs. Specifications with less error criteria are highlighted in green.
Source: Authors' calculations.

Appendix 4. Bayesian Averaging of Classical Estimates for 1-month, 12-month, 36-month, and 72-month yields

Following a BACE approach we present the estimates of Equation 34 for 1-month, 12-month, 36-month, and 72-month yields.

Table A4.1. Estimation results following a BACE approach

	1-month Treasury bond yield			12-month Treasury bond yield			36-month Treasury bond yield			72-month Treasury bond yield		
VARIABLES	Coefficient	SD	T	Coefficient	SD	T	Coefficient	SD	T	Coefficient	SD	T
TES bond yield$_{t-1}$	0.9577	0.0204	46.8956	0.9600	0.0111	86.3853	0.9606	0.0122	78.5342	0.9588	0.0145	66.1205
GDP$_t$				-0.0002	0.0007	-0.3647	-0.0003	0.0009	-0.3199	-0.0002	0.0008	-0.2141
GDP$_{t-1}$												
GDP growth$_t$				-0.0001	0.0026	-0.0372				0.0000	0.0029	0.0079
GDP growth$_{t-1}$												
5 Years CDS$_t$										0.0000	0.0001	-0.071
5 Years CDS$_{t-1}$										-0.0001	0.0004	-0.3065
Δ5 Years CDS$_t$							0.0001	0.0000	7.4728	0.0008	0.0001	8.5851
Δ5 Years CDS$_{t-1}$							-0.0040	0.0009	-4.7372	-0.0004	0.0001	-3.8716
CPI$_t$				0.0003	0.0003	1.0059	0.0003	0.0003	1.1966	0.0002	0.0002	0.7306
CPI$_{t-1}$				-0.0003	0.0003	-1.0177	-0.0003	0.0003	-1.2031	-0.0002	0.0002	-0.7311
Inflation rate$_t$	0.0580	0.0505	1.1495				0.2254	0.0652	3.4574	0.1214	0.0599	2.0253
Inflation rate$_{t-1}$												
MPR$_t$										0.0001	0.0033	0.0249
MPR$_{t-1}$										0.0000	0.0009	-0.0002
ΔMPR$_t$	0.9312	0.1886	4.9369	0.5689	0.0785	7.2441	0.0423	0.0446	0.9493	-0.0001	0.0184	-0.0073
ΔMPR$_{t-1}$	-0.8636	0.1932	-4.4706									
ΔInflation expectations$_t$				0.6801	0.1514	4.4908	0.9394	0.1568	5.993	0.9252	0.1788	5.1738
ΔInflation expectations$_{t-1}$				0.0000	0.0001	0.0002	0.0000	0.0002	0.0002	0.0000	0.0003	0.0003
(Intercept)	-0.0007	0.0014	-0.4843	0.0029	0.0011	2.6633	0.0024	0.0014	1.7458	0.0029	0.0014	2.0733
Observations		240			240			240			240	

The table summarizes the posterior estimates of the BACE model conducted. The left-hand side corresponds to the macroeconomic variables derived from the SYSMO model output variables. GDP denotes gross domestic product, CDS credit default swaps, and MPR monetary policy rate.

Source: Authors' calculations.

Appendix 5. Sampling algorithm for the confidence intervals of the impact of monetary policy surprises

This section details the procedure used to obtain the confidence intervals for the impact of unexpected monetary policy shocks on the TES yield curve.

1. Let the equation that relates the policy surprise to the change in the dynamic estimated factors in the GATSM be:

$$F_{t+h}^{r} - F_{t}^{r} = \alpha_{h}^{r} + \beta_{h}^{r} * \epsilon_{t} + \upsilon_{t+h}^{r}$$

F_{t+h}^{r} represents the three dynamic factors previously estimated for the yield curve *r* (COP or UVR) in period *t+h*, and ϵ_t is the policy shock.

2. The above equations are estimated, and the presence of autocorrelation in the errors $\upsilon_{t+h}^{r}$is validated. The last one is modeled with a VAR(p) autoregressive model, whose order p denotes the degree of autocorrelation in the residuals.

3. The errors of the step 1 model are sampled, and the coefficients $\{\alpha_{h}^{r},\beta_{h}^{r}\}$ p are used to obtain a new series of dynamic factors spreads $\{F_{t+h}^{r} - F_{t}^{r}\}_{boot}$. In case autocorrelation was detected in the estimation of step 1, the sampling is performed on the errors of the VAR(p) model. The coefficients of this model are also considered for generating the series of interest.

4. The series generated in the previous step is taken, and the equation relating these dynamic factor differentials to the policy surprise is estimated:

$$\{F_{t+h}^{r} - F_{t}^{r}\}_{boot} = \alpha_{h}^{1r} + \beta_{h}^{1r} * \epsilon_{t} + \nu_{t+h}^{1r}$$

5. The coefficient β_{h}^{1r} is stored.

6. Steps 1-5 are executed 1,000 times, thus obtaining a series of $\{\beta_{h}^{1r}, \ldots, \beta_{h}^{1000r}\}$.

7. For each maturity *n*, the response in yields is calculated using the coefficients estimated in step 1:

$$IRF_{h}^{r} = Y_{h,e}^{r} - Y_{h}^{r} = B_{n} * \beta_{h}^{r} * \epsilon_{t}$$

8. 1,000 responses in yields are calculated using the coefficients estimated in steps 4-6:

$$IRF_{h,boot}^{r} = Y_{h,e}^{r} - Y_{h}^{r} = B_{n} * \beta_{h}^{1000,r} * \epsilon_{t}$$

9. The confidence intervals of the response in the performance at term n are calculated as the 2.5% and 97.5% percentile of the responses $\{IRF_{h,boot}^{r}\}$.

This procedure is repeated for the maturities of interest in the two yield curves (COP and UVR) and each of the days of the post-shock horizon (between 1 and 90 days ahead).