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Pricing the exotic: Path-dependent American options with stochastic barriers*

Alejandro Rojas-Bernal† Mauricio Villamizar-Villegas‡

Abstract

We develop a novel pricing strategy that approximates the value of an American option with exotic features through a portfolio of European options with different maturities. Among our findings, we show that: (i) our model is numerically robust in pricing *plain vanilla* American options; (ii) the model matches observed bids and premiums of multidimensional options that integrate Ratchet, Asian, and Barrier characteristics; and (iii) our closed-form approximation allows for an analytical solution of the option's *greeks*, which characterize the sensitivity to various risk factors. Finally, we highlight that our estimation requires less than 1% of the computational time compared to other standard methods, such as Monte Carlo simulations.

Key Words: option pricing; exotic currency options; ratchet options; Asian options; American options; Barrier options; Weighted Time Value methodology; Least Squares Monte Carlo.

JEL Codes: C53, E58, G13

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Valoraciones exóticas:
El caso de opciones americanas con barreras estocásticas

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Resumen

En este documento proponemos una nueva metodología de valoración de opciones americanas con características exóticas mediante la valoración de un portafolio de opciones europeas con diverso vencimiento. Nuestros resultados muestran que: (i) la metodología es numéricamente robusta en la valoración de opciones americanas simples; (ii) las valoraciones del modelo corresponden a las ofertas y primas observadas en las subastas de un conjunto de opciones multidimensionales que integran elementos de opciones trinquete, asiáticas y barrera; y (iii) la forma cerrada de nuestra aproximación permite la derivación de una solución analítica para las griegas de la opción que caracterizan la exposición a diversos factores de riesgo. Finalmente, resaltamos que nuestro modelo requiere menos del 1% del tiempo de ejecución computacional comparado a otros métodos estándar como simulaciones de Monte Carlo.

JEL Classification: C53, E58, G13

Palabras Clave: valoración de opciones de divisas; opciones trinquete; opciones asiáticas; opciones barrera; ponderación del valor temporal; mínimos cuadrados de Monte Carlo

1 Introduction

To date, foreign exchange trading exceeds the volume of goods and services worldwide by almost 20-fold.[1] Moreover, total currency turnover (comprised of spot transactions, outright forwards, FX swaps, currency swaps, and FX options) increased by over 400% between 2001 and 2019. FX options, which are the main focus of our investigation, represent 5% of total trades, which amounts to $300 billion US dollars in one day (BIS, 2019). To put things in perspective, this amount is roughly the yearly GDP of an emerging market economy like Colombia, Egypt, Bangladesh, or Chile.

In the context of foreign exchange intervention, FX options have been employed by several central banks, mostly to smooth exchange rate volatility and to build-up or diminish international reserves (see Baillie and Osterberg, 1997; Villamizar-Villegas and Perez-Reyna, 2017; and Arango-Lozano et al., 2020). Such is the case of Mexico during 1996-2001, Colombia during 2002-2016, Australia in 1998, and Chile in 2008 (see Archer, 2005). The FX intervention literature acknowledges several attributes of currency options. First, similar to forward contracts, shorting options requires little-to-no immediate funding. Second, the structure and the transactions from the option can be tailored to different intervention mechanisms. For instance, trades can be anonymous or made public. Also, expiration maturities can be modified to different time horizons. Finally, triggering rules (i.e. to exercise the option) can be engineered so that a central bank sells (buys) foreign currency when its price is high (low).

Paradoxically, little is known about the valuation of most of these options, especially when they integrate an American style with exotic features (e.g. Ratchet, Asian, Barrier, and multidimensional options). Hence, our contribution is to shed light on this issue, by building a valuation strategy that allows us to approximate the value of an exotic American option through a portfolio of exotic European options. Specifically, we allow the weight of each European option to be related to its *time value*, which we define as the additional price that an investor is *willing to pay* over the option's intrinsic value, in order to compensate for the probability of a greater payoff, when exercised. As a result, exotic features can be more easily computed and approximated to their exact value. A shared caveat with other contending methodologies is that our model relies on heavy parametric assumptions regarding the distribution of the exchange rate.

Our methodology roots back to the classical risk-neutral measure of option pricing presented

[1] In April 2019, FX trading averaged $6.6 trillion US dollars per day, as reported in the 2019 Triennial Central Bank Survey of Foreign Exchange and Derivatives Market Activity (Bank for International Settlements-BIS).

in Black and Scholes (1973), and Merton (1973). More recently, the study that most closely relates to ours is Stozitzky (2015), who provides a valuation strategy for a similar class of FX options through Least Squares Monte Carlo (LSM) simulations. Our work is also related to Longstaff and Schwartz (2001) who derive the value of an American option through an optimal exercise strategy that maximizes the discounted expected value of the option's cash flow. However, when exotic features are introduced, several numerical issues arise in LSM. In particular: i) a high number of zero-value time periods turn up, ii) the method does not allow for possible option exercises within intermediate periods; and iii) the computational effort greatly increases (as a function of the data frequency).[2] Our method overcomes these issues in a manner that is not currently available in the literature, provided that we allow for a portfolio with a sufficient number of European options and that we have some information about the weighing function.

We recognize that our Weighted Time Value (WTV) methodology is not optimally derived, unlike LSM, which uses optimal control to estimate the expected future value from exercise. We do, however, validate the use of WTV by comparing it to the price of a *plain-vanilla* American option and find that our method outperforms LSM in terms of Mean Absolute Error (MAE) as well as Weighted Average Absolute Percentage Error (WAPE). Additionally, we also show that our valuation exhibits less volatile valuation trajectories. Finally, we show that our method requires only a fraction of the time to obtain results, compared to LSM estimations with high frequency data.

Our empirical estimation is based on the Colombian case during 2002-2012. We explore a particular intervention mechanism enacted by the Central Bank of Colombia (CBoC), entitled "Volatility Options", intended to curb exchange rate volatility. Specifically, FX options were triggered (auctioned) whenever the exchange rate vis-à-vis its past moving average exceeded a specific threshold. Once issued, options could only be exercised if the triggering rule was active in a given business day. Options expired after one calendar month. Our high frequency data, of proprietary nature, consist of the timing, amounts, bids, and resulting premia of each auction.

Our model yields some encouraging results. First, our estimations are closely comparable to the effective premium paid by market participants. Additionally, our closed-form approximation allows us to derive an analytical solution for the option's *greeks*, which characterize the option's sensitivity to various risk factors. In particular, we find that portfolio exposure lies more on the volatility rather than the level of the exchange rate. For this reason, as volatility increases, financial institutions are more prone to dynamically hedge their risk. Finally, we highlight that our estimation

[2]Possible explanations for these issues include misspecifications in the expectation function, limited sample simulation, or most likely, due to the lack of well-defined asymptotic properties.

requires less than 1% of the computational time compared to LSM.

We believe that our methodology can be useful for active practitioners that employ currency derivatives. In this sense, the WTV methodology provides a quick estimation of option prices for real-time users such as central banks, traders, and portfolio managers. Additionally, our method extends to a wide variety of option structures. This can allow central banks to evaluate (ex-ante) the expected option price, and the channels through which dynamic hedging operates in complex instruments. Finally, our method can be used to evaluate the cost-benefit analysis of foreign exchange interventions.

This paper proceeds as follows: In Section 2 we provide a brief literature review and highlight the different methodological approximations regarding option pricing. Section 3 describes the data and the characteristics of the options used by central banks when conducting foreign exchange intervention. Section 4 focuses on the two competing methods: the LSM algorithm and our proposed WTV methodology. In this section we present the intuition behind the WTV methodology and conduct numerical exercises that bear evidence that our method satisfies some ideal statistical properties and provides accurate estimations. In Section 5 we formally present our model, taking into account the exotic features of currency options employed by the CBoC. Finally, Section 6 presents the results of our model and Section 7 concludes.

2 Literature on Option Valuation

Option contracts give the holder or *long party* the right to buy or sell an underlying asset to another market participant or *short party*. Contracts in which the long side has the option to buy (sell) are known as call (put) options. Particularly, European *plain vanilla* options can be exercised at a fixed date of expiration, while American *plain vanilla* options can be exercised at any moment before or at the expiration; in both cases the exercise is given by a fixed *strike* price. The right of the option holder exists only after a premium has been paid upfront to the short party.

The valuation of these financial instruments has been widely treated in the literature. Broadly speaking, there are three different methodological approximations. The first one follows the Black and Scholes (1973) and Merton (1973) closed-form solution in finding the exact value of an option when some basic assumptions are satisfied. The second methodology depends on numerical methods, where the three most common are: (i) Monte Carlo Simulations (Boyle, 1977), (ii) recombining binomial trees (Cox, Ross, and Rubinstein, 1979), and (iii) finite difference methods (Hull and White, 1990). The third methodology is based on analytical approximations using the Edgeworth series expansion (Jarrow and Rudd, 1982).

Each one of the previous methodologies carries a tradeoff. While the first one does not require complex computational methods, its current use is limited to European options. Alternatively, the second method has the advantage of introducing specific contextual characteristics in the pricing of the option, such as allowing for multiple exercise dates before expiration, but is computationally time consuming. In fact, the binomial trees and finite difference approaches become impractical in options that depend on multiple factors. The last method overcomes the Gaussian distribution assumption of the first method, but requires market data in order to introduce higher independent moments of the probability distribution.

In this paper we center on the Black and Scholes (1973) and Merton (1973) closed-form approximation (**BSM**), but we contribute to the literature by including some of the more exotic features of currency options. Given the various extensions of BSM, we limit our attention to only those relevant for the construction of our model and the intuition behind it. For instance, Garman and Kohlhagen (1983) expand the BSM model for the valuation of European options on foreign currency. Grabbe (1983) adds on to the Garman and Kohlhagen approximation by introducing a stochastic behavior of domestic and foreign interest rates. Finally, Margrabe (1978) pioneers in developing a valuation model with a stochastic strike price. We thus follow Margabe's model when pricing Asian options when the *strike* constitutes a moving average price.[3]

3 Data and Context

The reasons for which a central bank might prefer FX intervention with the use of options over spot operations and other derivatives (e.g., forwards and FX swaps) are grounded in the way in which the hedging of these instruments occur. Namely, in forward and FX swap contracts, the risks that come along an open position can be completely hedged at the moment in which the instrument is acquired, generating a one time portfolio effect. In contrast, in option contracts the hedging strategies that are given by the option's greeks are dynamic and generate a constant portfolio balancing effect. For example, the risk that comes with a long forward position over one dollar could be hedged by selling one dollar, while in a call option, the risk over the foreign exchange rate could be partially hedged by selling the amount of dollars that is indicated by the option's delta, which is a measure that changes

[3] In regards to the latter case (Asian options with an average price), Kemna and Vorst (1990) develop a closed-form expression for a European option in which the price is given by a continuous geometric mean. Finally, Ritchken, Sankarasubramanian, and Vijh (1993) expand the Kemma and Vorst model for when the price is given by a discrete geometric mean.

over time. For this reason, when the foreign exchange is depreciating and the central bank sells call options, in order to hedge its position, the bearer requires to sell a changing amount of dollars during the days in which the option is active. These transactions occur on those days in which the foreign exchange rate is more volatile, which in turn helps in stemming depreciation pressures.

The design of FX options issued in Mexico and Colombia during 1996-2001 and 1999-2016, respectively, were fairly similar. Namely, once issued, the option holder could only exercise the option if the difference between the spot exchange rate and its moving average, generally spanning over 20 business days, exceeded an established threshold. Also, options expired exactly one calendar month after issuance. A key difference, however, was that in Mexico options were issued in pre-established dates with the purpose of building-up reserves. In Colombia, the purpose of these options was to stem exchange rate volatility, and options were issued (triggered) with the same rule as the one participants needed in order to exercise the option (see Mandèng, 2003, Canales et al., 2006, and Kuersteiner et al., 2018).[4]

We center our empirical application on the Colombia case, given the broader time frame in which Colombia intervened in the FX market with this particular class of options, the integration of exotic features (ratchet, Asian, barrier, and multidimensionality) in the structure of the option, and the availability of proprietary data that include the timing, amount, bids, and resulting premium of each option. Before turning our attention to the structure of these options, we note that the CBoC computes a daily reference exchange rate, entitled *Tasa Representativa de Mercado* or TRM. The TRM is simply the average of individual transactions (weighted by volume) that takes place on the previous trading day.[5] Also, in this paper we present the exchange rate in units of domestic currency per unit of foreign currency, or equivalently COP/USD.

Once the rule was triggered, the CBoC issued options through a clearing price auction. Financial intermediaries could present up to five bids without exceeding the authorized amount (almost always set at $180 million USD). At the end of the auction, all winners payed the same premium, i.e. a uniform clearing price. Finally, the strike price of the options corresponded to the TRM_t that is applicable at the day t of exercise.[6] This structural condition makes the options issued in Colombia path-dependent and, more precisely, an Asian option with a moving average strike, the

[4]For a review of the different channels through which FX intervention can be effective see Neely (2005), Menkhoff (2013), and Villamizar-Villegas and Perez-Reyna (2017). During this period, the CBoC also issued options to accumulate and diminish foreign exchange reserves, but due to the changing regulatory framework of these instruments, in this document we focus our attention solely to *volatility options*.

[5]The mean exchange rate in period t corresponds to the TRM_{t+1}.

[6]The regulatory framework of these options is found in document DODM-143 of the CBoC (2016).

strike being the weighted average of all transactions from the previous day.

Formally, the payoff of the option $\forall t : t \in [\tilde{t}, t^*]$, where $\tilde{t}$ denotes the starting point of the life of the option and t^* is the expiration date of the option is given by:

$$\text{For call options: } Max[S_t - TRM_t, 0] \;\; if \;\; TRM_t \geq \tfrac{1+f}{20} \textstyle\sum_{i=t-19}^{t} TRM_i$$
$$\text{For put options: } Max[TRM_t - S_t, 0] \;\; if \;\; TRM_t \leq \tfrac{1-f}{20} \textstyle\sum_{i=t-19}^{t} TRM_i$$

where S_t is the spot rate (COP/USD) at t, and f is the positive fixed percentage determined by the Board of Directors of the CBoC.[7]

4 Contending Valuation Methods

4.1 Least Squares Monte Carlo (LSM)

The LSM methodology assumes a complete probability space (Ω, F, P) and finite time horizon $[0, T]$, where Ω represents all possible states ω in a stochastic economy, and $\mathscr{F}$ is the sigma-algebra of events whose elements can be assigned probabilities $\mathscr{P}$. As time passes, the relevant price processes generate, at time t, the augmented filtration $F_t = \{\mathscr{F}_t; t \in [0, T]\}$. We assume henceforth the existence of a martingale measure $\mathscr{Q}_t$ that is guaranteed by a no-arbitrage condition.[8] Following Longstaff and Schwartz (2001), we represent the value of an American option using the *Snell Envelope*, according to which the value of an American option is given by an optimal control exercise strategy that maximizes the discounted value of the option cash flow. This maximum covers all stopping times with respect to the filtration F_t.

The LSM methodology provides a stopping rule that maximizes the value of the American option while taking into account a discrete number of exercise periods. This algorithm works backwards, starting from the last to the first period. In each one of the stopping periods t we are certain about the cash flow value of exercising the option, but we are unsure about the payoff in future stopping periods. The logic behind the algorithm is to estimate the expected future payoff using only the information available in F_t conditional on the trajectories in which a positive payoff at t is obtained. In other words, we are interested in finding the value of continuation $G(\omega; t_k)$ at t_k,

[7] This percentage threshold took values between 2% - 5% (see Kuersteiner et al., 2018 for a detailed description).

[8] We argue that any version of the *Efficient Market Hypothesis* is sufficient for any no-arbitrage condition to hold in the context of foreign exchange markets (see Fama (1970), Mussa (1976), and Meese and Rogoff (1983)). In particular, using Girsanov Theorem, with the filtration F_t this equivalent martingale measure or risk neutral probability at t, is such that the initial value of the derivative equals the present discounted expected value of the derivative final payoff.

which can be expressed as:

$$G(\omega;t_k) = \widetilde{E}_{\mathscr{Q}t}\left\{\sum_{i=k+1}^{T} exp\left(-\int_{t_k}^{t_i} r(\omega,s)ds\right) C(\omega,t_i;t_k,T)|\mathscr{F}_{t_k}, C(\omega,t_k;t_k,T) > 0\right\}.$$

Where $r(\omega,s)$ is the stochastic riskless discount rate, $C(\omega,t_i;t_k,T)$ are the remaining cash flows generated by the option, and $\widetilde{E}_{\mathscr{Q}t}$ is the expected value function with respect to the risk-neutral measure, the filtration $\mathscr{F}_{t_k}$, and conditional to those trajectories in which a positive payoff $C(\omega,t_k;t_k,T) > 0$ is obtained if exercised at t_k.

The optimal exercise problem in the LSM algorithm is reduced to comparing the immediate exercise value $C(\omega,t_k;t_k,T)$ with respect to the conditional expectation $G(\omega,t_k)$. An option is worth exercising as long as $C(\omega,t_k;t_k,T) \geq G(\omega,t_k)$; this means that at t_k the American option is exercised whenever its current payoff is greater than its future expected payoff. In estimating the conditional expected value $G(\omega,t_k)$, Longstaff and Schwartz (2001) use a linear approximation (OLS) in which the covariates in X are limited by the information available in the filtration $\mathscr{F}_{t_k}$. The authors show that the results obtained by the linear function of the three first polynomials of the spot price in ω at t_k are virtually identical to those given by other basis functions.[9]

Following this result we implement our algorithm with the first three polynomial degrees of the spot price in ω at t_k as control variables, and as dependent variable we use the conditional expected value of future payoffs $G(\omega,t_k)$ discounted with the constant risk-free interest rate r_d. This algorithm includes the mean of the price process for the last day and the average of the last twenty days as proxies of the TRM_t and of its 20-day moving average. We also took into account the condition to exercise the option (i.e. that the TRM_t must be greater or equal than its 20-day moving average by a percentage f).

For the option auctioned in February 12 of 2009, in Figure 1 we show the price trajectories given by the LSM algorithm for a volatility between 1% and 20%, and for fixed percentages f =1%, 2% and 3%. As expected, exchange rate volatility has a positive effect on the price of the option, while the percentage f has a negative effect.

Once we study the optimal exercise matrix for all call and put options auctioned by the CBoC we find that the simulated prices of the options are given by corner solutions: either the option is to be exercised immediately or it is to be kept and exercised at the last possible moment. This results is

[9] Longstaff and Schwartz consider the first three Laguerre polynomials, Hermite polynomials, and trigonometric functions as basis functions in the OLS approximation.

sub-optimal due to the fact that 38.2% of the notional amount of the call options and 49.9% of the notional amount of the put options were exercised on days after the day of the auction. In Figure 2 we depict a density histogram for the period in which the option is exercised according to each of the simulations of the LSM methodology. From this figure it is manifest that the value of the option given by the LSM algorithm takes into account only some exercise periods. Specifically, the option's premium is heavily explained by the exercise on the first day, and it completely ignores all the intermediate periods, even thought the exercise in these periods is possible and a Monte Carlo simulation should assign some expected value on these days. For this reason, once we simulate the value of these options for every trading day with the LSM algorithm we find that the call and put options have a predicted value of zero in 65.2%, and 69.8% of cases, respectively, as shown in Figures 3 and 4. We refer to these episodes as *absolute unawareness*.

For the absolute unawareness of LSM of all intermediate periods we have three potential explanations: (i) the polynomial basis function for LSM is not robust, although this would contradict the results in Longstaff and Schwartz (2001), (ii) considering one thousand trajectories per simulation is not enough to guarantee the asymptotic properties of LSM, or (iii) the higher dimensional structure of the exotic option thwarts the uniform convergence of the basis function that is needed to guarantee the asymptotic properties of LSM.[10]

4.2 Weighted Time Value (WTV)

The value of a European option is given by two components. First, the *intrinsic value* of an option is the payoff obtained if the option could be exercised immediately. Specifically, the *intrinsic value* of a call and put option is given by $Max(S-K,0)$ and $Max(K-S,0)$, respectively, where S denotes the spot price and K is the strike price. Second, the *time value* of an option is given by the difference between the value of the option and its *intrinsic value*. The value of an option is hence equal to the *intrinsic value* plus its *time value*.

The literature on the behavior of the *time value* is primarily focused on the decay of this measure and how the interaction with other variables such as price and volatility alters this decay. These

[10]The proof of the asymptotic properties of LSM in Longstaff and Schwartz (2001) *is limited to one-dimensional settings*, while the structural dependence of our options to the spot price, the TRM_t and the average of order 20 of the TRM_t generates a multidimensional setting for which the authors only *conjecture that similar results can be obtained for higher-dimensional problems by finding conditions under which uniform convergence occurs*. The presumable lack of uniform convergence in the valuation of these options is probably due to the choice of the basis function; in particular, Longstaff and Schwartz (2001) proof the consistency of LSM in a one-dimensional setting employing the property of uniform convergence of the Laguerre polynomials, the robustness of other basis function is a consequence of their numerical tests.

studies are either empirical using the effective price of traded options (Brozik, 2014; McKeon, 2017), or theoretical through simulations (Tannous and Lee-Sing, 2008) or a BSM model approximation (Emery et al., 2008). The following exercise is made with a *ceteris paribus* assumption in a BSM modeling approach.

Figures 5.A and 5.B show the *time value* for call and put options *at-the-money* with positive and negative interest rates, respectively. As can be seen, for a certain maturity, the *time value* of a call option is greater when the interest rate is positive, and the *time value* of a put option is greater when the interest rate is negative. Similarly, Figures 5.C, 5.D, 5.E, and 5.F, respectively, display for different spot prices; the *time value* for *in-the-money* call and put options with positive and negative interest rate. The behaviour of these *time values* reflects the idea that with a positive interest rate, the risk neutral measure layoffs an expected higher forward price, which means an anticipation of a rising *intrinsic value* for call options and a diminishing *intrinsic value* for put options. Conversely, a negative interest rate suggests an expected lower forward price, which means a shrinkage of the *intrinsic value* in call options and an expansion of the *intrinsic value* in put options.

Following this results, intuitively, the *time value* can be thought of as the additional price that an investor pays over the current *intrinsic value*, in order to compensate for the probability that its value increases at expiration. Therefore, the only difference between options with similar *strikes* and different maturities is in their *time values*. Namely, these *time values* can be thought of as portfolio weights. Hence, we proceed by showing that a time-value weighted portfolio of European options is able to replicate the value of an American option. Laprise et al. (2006) develop a intuition analogous to ours in which the value of an American option is approximated by pricing a portfolio of European call options; the difference in our approach is given by the role that the *time value* plays in the design of this portfolio.

For simplicity, we are trying to replicate the value of an American call option over one dollar with a portfolio of N European options with similar *strike* (also over one dollar), each with a different expiration date not greater than the expiration of the American option. Let $g(tv_i)$ be a function of the time value tv_i of the European option i that has a value υ_i given by BSM. Consequently, the approximate value of the American option ($a\upsilon$) given by a portfolio of N European options with the same notional amount and different expiration dates can be approximated by:

$$a\upsilon \approx \sum_{i=1}^{N} m\, g(tv_i) \left[\sum_{i=1}^{N} g(tv_i) \right]^{-1} \upsilon_i$$

where $\beta_i = m\, g(tv_i)\, [\sum_{i=1}^{N} g(tv_i)]^{-1}$ is the weight for the price of the option with value υ_i, and m is the aggregate notional amount on the portfolio needed to equal the value of the American option.

If $m = 1$, the weights are such that $\sum_{i=1}^{N} \beta_i = 1$, and we end up with a portfolio of N European options that has an aggregate amount equal to the aggregate amount of the American option. The infinite exercising trajectories of the American option include all trajectories of the N European options in the portfolio in such a way that the value of the American option cannot be inferior to the value of each of the N European options ($a\upsilon \geq \upsilon_i \ \ \forall i$). Also, for a given price path, the American option includes the optimal period of exercise, while the portfolio of European options only approximately includes this optimal exercise through those options that are close to the optimal exercise. Let's assume that one of the European options has a maturity that corresponds to the optimal exercise of the American option. In this case, if the whole weight of the portfolio is given to this option then the aggregate notional amount needed to equal the American option is $m = 1$. In any other portfolio that has a linear combination of the European options, the aggregate notional amount required to compensate for the set of *sub-optimal* European options must be one in which $m > 1$.

As our study from the path-dependent options auctioned by the CBoC and the central bank of Mexico (*Banxico*) has led us to believe, the advantage of using WTV over LSM, as shown in the following sections, is that it is a better predictor of the value of some exotic American options. The main reason is that the LSM algorithm in some multidimensional options with exotic features, ignores the payoff for some periods and events that despite their unlikeliness, should hold some positive expected value. This problem can be overcome through an appropriate calibration of the WTV.

4.3 LSM versus WTV

In order to validate the WTV methodology, we use the BSM model as benchmark comparison. Note that in currency call options with a risk-neutral valuation and a domestic interest rate greater than the foreign interest rate, the price of a European call option is the same as the price of an American call option (see Capinski and Zastawniak 2011 and Hull 2015). Alternatively, for put options the benchmark valuation is given by LSM. We thus conjecture that for call options the value of the BSM, WTV, and LSM methodologies are similar, while for put options the value of the WTV, and the LSM algorithm are comparable.

In order to test these inferences we study the behaviour of the weights $g(tv_i)$ through the following Restricted Ordinary Least Squares (ROLS) regression with no constant in *plain vanilla* call and put options:

$$a\upsilon(S_i,T) = \sum_{j=1}^{N} \beta_j \upsilon\left(S_i, j\,\frac{T}{N}\right) + e_i,$$

where $a\upsilon(S_i,T)$ is the value of the American option with maturity T and a present spot price S_i given by BSM in call options and LSM in put options, the term $\upsilon\left(S_i, j\frac{T}{N}\right)$ is the value of the European option with a present spot price S_i given by BSM, an expiration date of $j\frac{T}{N}$, and e_i is the regression error. The restrictions imposed on this regression are such that $\forall j \in [1,N]: \ \beta_j > 0$ and $\sum_{j=1}^{N} \beta_j = m$.

In Figures 6 and 7 we observe, respectively, values for β_j estimated with the ROLS regression for call and put options with different prices. As shown, the prices given by the WTV methodology are similar to the prices given by the BSM model in call options and the LSM simulations in put options. From these figures we can see that a greater time value is directly related with the weight of the European option in the aggregate portfolio.[11]

In Figure 8 we see the value of a European call option given by the BSM model and the value of an American call option given by both the LSM approximation and several weights used in the WTV portfolio.[12] We find that all the weights considered in the WTV portfolio have a more stable path than the LSM algorithm around the benchmark value (given by the BSM model).[13] Particularly, the WTV with the function $exp(2tv_i)/\sum_{i=1}^{N} exp(2tv_i)$ has a remarkable behaviour, with a mean absolute error (MAE) and a weighted average absolute percentage error (WAPE) that are 72% of the MAE and WAPE of the LSM algorithm (Table 1).[14]

[11] In call and put options the aggregate notional amount of the WTV portfolio needed to minimize the sum of squared residuals between its price and the American option price are $m = 1.0179$ and $m = 1.0305$, respectively.

[12] Particularly we look at the WTV given by: i) a linear weighting of the *time value* ($tv_i/\sum_{i=1}^{N} tv_i$); ii) a simple average of the value of the options in the portfolio; iii) the value given by allocating all the weight to the European option with the greatest *time value*; and iv) the value given by the allocation of weights according to the formula $exp(2tv_i)/\sum_{i=1}^{N} exp(2tv_i)$. These portfolios are built with the previously found aggregate notional amount for call and put options of $m = 1.0179$ and $m = 1.0305$, respectively. In order to avoid negative weights (when there is a negative *time value*) we use the *time values* after netting out the minimum *time value* of the set of possible European options, this is $tv_i - Min(tv_i)$.

[13] The behaviour of the LSM prediction is consistent with the findings of Longstaff and Schwartz (2001), according to which the LSM prediction will always be lower or equal to the true value of the option. The large deviations of the LSM prediction to the true value given by BSM appear on different spot prices on every simulation.

[14] Given that certain measures such as the mean absolute percentage error (MAPE) can be highly dependent in values *out-of-the-money* where the option premium is low and the MAPE can be magnified, we also consider a weighted average absolute of the mean percentage error (WAPE) in which the weights are given by the values of the BSM model; as $\sum_{i=1}^{N}\left(\frac{BSM_i}{\sum_{i=1}^{N} BSM_i}\right)|\frac{\hat{C}_i - BSM_i}{BSM_i}|$.

Table 1: LSM against WTV in call options:MAE and WAPE

	Mean Absolute Error	Weighted Average Absolute Percentage Error
LSM	0.2226	0.0141
WTV	0.1603	0.0104

SOURCE: Authors' calculations.

In Figure 9 we see the value of a European put option given by the BSM model, and the value of an American put option given by both the LSM methodology and several weightings in the WTV portfolio. For put options it can be seen that all the prices obtained with the WTV methodology move around the benchmark value given by LSM. Particularly, the WTV given by a portfolio with linear weighting shows a behaviour similar to the LSM algorithm. As expected, the value given by the BSM model is lower due to the fact that it doesn't take into account all feasible exercises of the American option before its maturity.

Consequently, provided that we allow for a portfolio with a sufficient number of European options, these findings suggest that the WTV methodology provides accurate estimations for the generally unknown true value of an American option. For this reason, WTV complements the set of numerical methodologies that are currently used for the estimation of an American option premium.

5 The Model

5.1 Components of Exotic Options

Some studies have attempted to establish the value of exotic American path-dependent options, such as the ones employed by central banks. Among the few, Stozitzky (2015) uses the LSM algorithm but modified to include a Merton (1976) mixed Jump-Diffusion model. Stozitzky finds that options issued by the CBoC were purchased at a lower price than the theoretical price, and attributes his finding to a low level of liquidity or a lack of market awareness. We note, however, that Stozitzky simulates trajectories of average daily exchange rate (TRM_t) and not the spot rate at each point during the day.

The intuition behind our valuation model is given by an eclectic approximation to exotic options. This means that in order to value the options used by the CBoC, we simultaneously take into account the different exotic elements that are comprised in these instruments. In particular, we take into account: (i) Ratchet options, (ii) Asian options with an average strike, and (iii) options with stochastic barriers.

These components are further described as follows: Ratchet (or strike-reset options) are a class of European options with an explicit rule for setting the strike price. For example, consider a ratchet portfolio with N options and reset dates $\tau, 2\tau, \ldots, N\tau$. On any one of these reset days, each option can be exercised with a strike price K_n with $n \in [1,N]$.[15] In other words, Ratchet options are a set consisting of a regular option and $N-1$ forward options (see Liao and Wang, 2003; Hull, 2015).

In turn, Asian options are path-dependent options in which either the underlying variable price or the strike price is given by an average. In the case of average price call options, the strike is fixed and the payoff when exercised is given by $Max[S_t^{Ave} - K, 0]$. For average strike call options, the payoff is $Max[S_t - S_t^{Ave}, 0]$.[16]

Finally, in Barrier options the final payoff depends on whether the price trajectory crosses a specific threshold. They are known in the literature as *knock-in* and *knock-out* barrier options that allow or restrict the possibility of exercise at a given period, respectively (see Derman and Kani, 1996 and 1997; Hull, 2015).

5.2 Options Issued by the Central Bank of Colombia

Consider first a simplified version of the options auctioned by the central bank in which there is no exercise condition. Following Section 3, the strike price is given by the TRM_t. Therefore, the value of these options can be approximated by assuming that each option is a ratchet option composed by a portfolio of Asian average strike options in which the strike resets each day, according to the behaviour of the exchange rate.

We proceed by using the WTV methodology in order to guarantee that the notional amount of this ratchet option is similar to that of the original exotic option. Therefore, the value of a call option that can be exercised during N business days is roughly given by $C = \sum_{n=1}^{N} \kappa_n c_n$, where κ_n is the weight factor assigned with the WTV methodology for each one of the options with premium c_n that conforms the portfolio.

Note that, as exhibited in section 4.3 for *plain vanilla* American options, we would expect a moving time-value that yields a non-constant weight of κ_n. This moving time value would reflect the

[15] The strike price K_n can be deterministic or stochastic. If stochastic, the option becomes a path-dependent option.
[16] See Kemna and Vorst (1990), and Ritchken et al. (1993).

additional price that an investor is willing to pay for the probability of future changes in the *intrinsic value*. Nonetheless, in our case the strike price (TRM_t) is reset each day with the market information from the previous day in such a way that any previous rise in the *intrinsic value* is almost completely offset by the new strike. Thus, we find reasonable to assume a constant time value for these options (as the uncertainty is eliminated each day) in such a way that $\kappa_n = N^{-1}$ and $C = N^{-1}\sum_{n=1}^{N} c_n$. We validate this assumption by finding for those options that were auctioned by the CBoC the κ_n that would be given by a linear WTV in which the *time value* is found by assuming that the current and known *intrinsic value* is applicable to all the c_n. In Figure 10 we display the average κ_n for the call and put options auctioned by the CBoC using the model developed in this section, as it can be seen, the assumption of $\kappa_n = N^{-1}$ is a good approximation.

If market participants expect that the intervention will be effective in stemming FX volatility, the assumption of a constant κ_n is objectionable only if effectiveness varies across time. Put differently, if agents expect a greater (lower) reduction in volatility during the final days of the option, the κ_n should decrease (increase) with n. Consequently, our assumption of a constant κ_n takes for granted the expectation of an homogeneous effect of intervention.[17]

Note that the condition to exercise the option depends on the value of the TRM_t with respect to its 20-day moving average. Consequently, the value of the ratchet option with an average strike and a simplified barrier, is given by:

$$C = N^{-1}\sum_{n=1}^{N} P\left[TRM_n \geq \frac{(1+f)}{20}\sum_{i=n-19}^{n} TRM_i\right] c_n. \tag{1}$$

Before constructing our model, we make the following standard assumptions commonly found in the BSM framework:

a) In each moment of time there is an unique price for every asset and financial instrument.

b) Short term domestic and foreign interest rates are known and constant.[18]

c) There are no market frictions (no transaction costs). Also, infinite divisibility and infinite liquidity are allowed, and short selling is possible without penalties.

d) The distribution of the foreign exchange rate is log-normal. As a consequence, the returns on

[17] We note that if agents believe that options will be effective in stemming FX volatility, it is possible that their portfolio strategy would reflect this belief, and their expectations will become self-fulfilling. As a consequence, the probability of exercising these instruments would be diminished.

[18] This assumption could be put aside defining stochastic foreign and domestic interest rates as in Grabbe (1983).

foreign currency are normally distributed.

e) There is continuous trading.[19]

First, we define $H(t) = lnS(t)$. As stated, we assume that the foreign exchange rate follows a classical geometric Brownian motion with a stochastic process given by $dS = \mu S dt + \sigma S dz$, where μ and σ respectively represent the expected return and the constant volatility or standard deviation of the foreign currency; and dz is a Wiener process in continuous time.[20]

Following Itô's Lemma we know that $dH = \frac{\partial H}{\partial t}dt + \frac{\partial H}{\partial S}dS + \frac{1}{2}\frac{\partial^2 H}{\partial S^2}dS^2$. Given that $\frac{\partial H}{\partial t} = 0$, $\frac{\partial H}{\partial S} = \frac{1}{S}$ and $\frac{\partial^2 H}{\partial S^2} = -\frac{1}{S^2}$ and after replacing dS and $dS^2 = \sigma^2 S^2 dt$, it follows that $dH = \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dz$. Also, a consequence of the normal distribution of the Wiener Process is that $lnS(t^*) - lnS(t) \sim \phi\left[\left(\mu - \frac{\sigma^2}{2}\right)(t^* - t); \sigma^2(t^* - t)\right]$. Hence, $S(t^*)$ has the following log-normal distribution:[21]

$$S(t^*) \sim \Lambda\left[lnS(t) + \left(\mu - \frac{\sigma^2}{2}\right)(t^* - t); \sigma^2(t^* - t)\right]. \tag{2}$$

Assuming that the domestic and the foreign interest rates are known and constant, and that the market is complete, there exists a unique martingale that guarantees a unique risk-neutral probability[22]. That is, from the Uncovered Interest Parity (UIP) condition it follows that the continuous risk-neutral return for a foreign currency is the expected depreciation (i.e, $\mu = r_d - r_f$, where r_d and r_f denote domestic and foreign interest rates). Equation (2) can then be restated as:

$$S(t^*) \sim \Lambda\left[lnS(t) + \alpha(t^* - t); \sigma^2(t^* - t)\right], \tag{3}$$

where $\alpha = \left(r_d - r_f - \frac{\sigma^2}{2}\right)$.

To compute a proxy of the distribution of the TRM_t and its 20-day moving average, we use

[19] We ignore before and after hours of trading. Note that this assumption can be overcome with a Merton's mixed Jump-Diffusion Model, but in the sake of parsimony we develop a simplified framework.

[20] The assumption of constant volatility can be left aside by using models with stochastic volatility (Hull and White (1987)).

[21] Here we follow the notation of Aitchison and Brown (1963) where X is a variable such that $Y = \log X$ is normally distributed with mean μ and variance σ^2. It follows that X is log-normal and write $X \sim \Lambda(\mu, \sigma^2)$, and $Y \sim \phi(\mu, \sigma^2)$. From here we get a first restriction for $X : 0 < X < \infty$. From $Y = \log X$ we can establish a relationship between the distributions of X and Y: $\Lambda(x) = \phi(\log x)$ for $x > 0$. Hence $\Lambda(x) = 0$ for $x \leq 0$, and $d\Lambda(x) = \left[\exp\left(-\frac{\log x - \mu}{2\sigma^2}\right)\right] / \left[x\sigma\sqrt{2\pi}\right] dx$ for $x > 0$.

[22] This implies the assumption that the market is complete.

a discrete geometric mean defined as $G(t_k,t_p) = [S(t_{k+1})S(t_{k+2})...S(t_p)]^{1/p-k}$.[23] This assumption is reasonable because, even though in our data the arithmetic average is always greater than the geometric average, the largest difference between the arithmetic and the geometric mean for one day is 0.01%, and for 20 days is 0.10%.[24]

Following Ritchken et al. (1993), we define a discrete geometric average of order j at time t with known and unknown components as:

$$\begin{aligned} G_{i|j} &= G(t_{p_i - j(p_i-k_i)}, t_{p_i}; t) \\ &= G(t_{p_i(1-j)+jk_i}, t_{b_{i|j}}; t)^{\frac{b_{i|j} - p_i(1-j) - jk_i}{j(p_i-k_i)}} \widetilde{E}_t \left[G(t_{b_{i|j}}, t_{p_i}; t)^{\frac{p_i - b_{i|j}}{j(p_i-k_i)}} \right] \\ &= e^{y_{i|j}} \widetilde{E}_t \left[G(t_{b_{i|j}}, t_{p_i}; t)^{\frac{p_i - b_{i|j}}{j(p_i-k_i)}} \right]. \end{aligned}$$

Where $0 = t_0 \leq t_{p_i(1-j)+jk_i} \leq t_{k_i} < ... < t_{p_i} < ... < t_{q_i} \leq t_n = T$ with $t_{b_{i|j}} \leq t < t_{b_{i|j}+1}$ if $t_{p_i(1-j)+jk_i+1} \leq t$ and $t_{b_{i|j}} = t_{p_i(1-j)+jk_i}$ otherwise, and $\Delta t = t_i - t_{i-1} = \frac{T}{n}\ \ \forall i \in [1,n]$.[25]

In this framework, t_0 is the first sequential term needed to compute the moving average of order j; t_n corresponds to the last term for which we need information; and t_{q_i} is the moment in which the option i that conforms the portfolio of the ratchet option may be exercised.[26] The expectation $\widetilde{E}_t$ is taken with respect to the equivalent risk neutral measure that allow us to price all securities. This definition is general because if $t < t_{p_i - j(p_i-k_i)+1}$, then the geometric mean is fully stochastic and $y_{i|j} = 0$. Also, if $t_{p_i} \leq t$, then the geometric mean is known, and hence $G_{i|j} = e^{y_{i|j}}$.

Assuming that the returns on the foreign currency $H(t^*) = lnS(t^*)$ are correlated in common time periods in such a way that $Cov(H(t_1), H(t_2 - t_1)) = 0$ and $Cov(H(t_1), H(t_2)) = Var(H(t_1)) \forall t_1 \leq t_2$[27] we have that:

[23] For the computation of the 20-day moving average, we use a geometric average twenty times longer than the TRM_t. Note that the geometric mean is less than or equal than the arithmetic mean (Beckenback and Bellman, 1971). Our selection of the geometric average allow us to use the properties of the lognormal distribution.

[24] This difference is greatly influenced by the market volatility, e.g., in our data 6 out of 10 observations that overpass the 0.005% percentage difference between the arithmetic and the geometric average are from 2008.

[25] Also $y_{i|j} = \frac{b_{i|j} - p_i(1-j) - jk_i}{j(p_i-k_i)} Ln\left(G(t_{p_i(1-j)+jk_i}, t_{b_{i|j}}; t)\right)$.

[26] For simplicity we assume that this happens at the last moment of the day, thus $t_{q_i} - t_{p_i} = t_{p_{i+1}} - t_{p_i}\ \ \forall i \in [1,n]$.

[27] In order to validate this assumption we generated 2 million possible combinations of t_0, t_1, and t_2, and tested whether the returns on the foreign exchange between t_0 and t_1, and t_1 and t_2 where independent. We found that the correlation coefficient between these series is $10,78\%$.

$$G(t_{b_{i|j}},t_{p_i};t)\sim\Lambda\left[\varphi'\begin{pmatrix} lnS(t)+\alpha(t_{b_{i|j}+1}-t)\\ \vdots \\ lnS(t)+\alpha(t_{p_i}-t)\end{pmatrix};\sigma^2\varphi'\overbrace{\begin{pmatrix}(t_{b_{i|j}+1}-t) & (t_{b_{i|j}+1}-t) & \cdots & (t_{b_{i|j}+1}-t)\\ (t_{b_{i|j}+1}-t) & (t_{b_{i|j}+2}-t) & \cdots & (t_{b_{i|j}+2}-t)\\ \vdots & \vdots & \ddots & \vdots\\ (t_{b_{i|j}+1}-t) & (t_{b_{i|j}+2}-t) & \cdots & (t_{p_i}-t)\end{pmatrix}}^{V_{b_{i|j},p_i}}\varphi\right]$$

where $\varphi'=\left[\frac{1}{p_i-b_{i|j}}\cdots\frac{1}{p_i-b_{i|j}}\right]_{(1,p_i-b_{i|j})}=\dfrac{1}{p_i-b_{i|j}}i'$.[28] It follows that:[29]

$$G(t_{b_{i|j}},t_{p_i};t)\sim\Lambda\left[lnS(t)+\frac{\alpha}{p_i-b_{i|j}}\overbrace{\sum_{f=1}^{p_i-b_{i|j}}(t_{b_{i|j}+f}-t)}^{(p_i-b_{i|j})\Psi_{i|j}};\frac{\sigma^2}{p_i-b_{i|j}}\Upsilon_{i|j}\right]=\Lambda\left[lnS(t)+\alpha\Psi_{i|j};\frac{\sigma^2}{p_i-b_{i|j}}\Upsilon_{i|j}\right].$$

As a consequence:[30]

$$G_{i|j}\sim\Lambda\left[\overbrace{y_{i|j}+\frac{p_i-b_{i|j}}{j\,(p_i-k_i)}\left(lnS(t)+\alpha\Psi_{i|j}\right)}^{\theta_{i|j}};\overbrace{\frac{\sigma^2\,(p_i-b_{i|j})}{j^2\,(p_i-k_i)^2}\Upsilon_{i|j}}^{\vartheta^2_{i|j}}\right].$$

Define $(1+f)=e^g$. We then have that $(1+f)G_{i|j}\sim\Lambda\left[\beta_{i|j};\vartheta^2_{i|j}\right]$ where $\beta_{i|j}=g+\theta_{i|j}$. In order to introduce the exercise condition into our valuation we define the probability $P\left(\frac{G_{i|j=1}}{(1+f)G_{i|j=20}}>1\right)$, where $G_{i|j=1}\left[(1+f)G_{i|j=20}\right]^{-1}\sim\Lambda\left[\theta_{i|j=1}-\beta_{i|j=20};\vartheta^2_{i|j=1}+\vartheta^2_{i|j=20}\right]$.[31] Recall that we are inter-

[28] If $\mathbf{X}$ is multivariate lognormal with mean $\boldsymbol{\mu}$, variance and covariance matrix $\mathbf{V}$ and a vector $\mathbf{b}$ of constants with transpose **b'**, then the product $c\prod_j X_j^{b_j}$ is $\Lambda(a+\mathbf{b'}\boldsymbol{\mu},\mathbf{b'Vb})$, where $c=e^a$ is a positive constant.

[29] Where $\sum_{f=1}^{p_i-b_{i|j}}\left(t_{b_{i|j}+f}-t\right)=\left(p_i-b_{i|j}\right)\left(t_{b_{i|j}+1}-t\right)+\Delta t\sum_{f=1}^{p_i-b_{i|j}-1}f$. Given that $\sum_{f=1}^{n}f=\frac{n(n+1)}{2}$ we have that $\sum_{f=1}^{p_i-b_{i|j}}(t_{b_{i|j}+f}-t)=\left(p_i-b_{i|j}\right)\left[(t_{b_{i|j}+1}-t)+\frac{T}{n}\left\{\frac{p_i-b_{i|j}-1}{2}\right\}\right]=\left(p_i-b_{i|j}\right)\Psi_{i|j}$. Also $i'V_{b_{i|j},p_i}i=(t_{b_{i|j}+1}-t)(p_i-b_{i|j})^2+\Delta t\left[(p_i-b_{i|j})\sum_{f=1}^{p_i-b_{i|j}-1}f-\sum_{f=1}^{p_i-b_{i|j}-1}f^2+\sum_{s=1}^{p_i-b_{i|j}-2}\sum_{f=1}^{s}f\right]$. Given that $\sum_{f=1}^{n}f^2=\frac{n(n+1)(2n+1)}{6}$ and $\sum_{s=1}^{n}\sum_{f}^{s}f=\frac{n(n+1)(n+2)}{6}$ we have that $i'V_{b_{i|j},p_i}i=(t_{b_{i|j}+1}-t)(p_i-b_{i|j})^2+\frac{T}{3N}(p_i-b_{i|j})(p_i-b_{i|j}-1)(p_i-b_{i|j}-0.5)=(p_i-b_{i|j})\Upsilon_{i|j}$.

[30] If $X\sim\Lambda(\mu,\sigma^2)$ and b and c are constants, where $c>0$ (say $c=e^a$), then $cX^b\sim\Lambda(a+b\mu,b^2\sigma^2)$.

[31] If X_1 and X_2 are independent Λ-variates, then the product X_1X_2 is also a Λ-variate. More precisely, if $X_1\sim\Lambda(\mu_1,\sigma_1^2)$

ested in the call option that expires at q_i and has a price given by:

$$c_i = e^{-r_d(q_i-t)} P\left(G_{i|j=1}\left[(1+f)\,G_{i|j=20}\right]^{-1} > 1\right) \widetilde{E}_t\, max\left[S\left(t_{q_i}\right) - G\left(t_{k_i}, t_{p_i}\right), 0\right]$$
$$= e^{-r_d(q_i-t)}\left[1-H_i\right]\widetilde{E}_t\, max\left[S\left(t_{q_i}\right) - G_{i|j=1}, 0\right]$$

where $H_i = \bar{H}\left(1; \theta_{i|j=1} - \beta_{i|j=20}; \vartheta^2_{i|j=1} + \vartheta^2_{i|j=20}\right)$, $\bar{H}$ is the log-normal cumulative distribution function for the value 1 given $\theta_{i|j=1} - \beta_{i|j=20}$ and $\vartheta^2_{i|j=1} + \vartheta^2_{i|j=20}$ as the mean and variance of the associated distribution, respectively.

The resulting option is similar to the one presented in (Margrabe, 1978), with two exchangeable assets prices $S\left(t_{q_i}\right)$ and $G_{i|j=1}$. We thus use $G_{i|j=1}$ as numeraire and obtain that $c_i = e^{-r_d(q_i-t)}\left[1-H_i\right]\widetilde{E}_t\left[G_{i|j=1}\right]\widetilde{E}_t\, max\left[\frac{S\left(t_{q_i}\right)}{G_{i|j=1}} - 1, 0\right]$.

From equation (3) it follows that:

$$\frac{S(t_{q_i})}{G_{i|j=1}} \sim \Lambda\left[\frac{b_{i|1}-k_i}{p_i-k_i} lnS(t) + \alpha\overbrace{\left[\left(t_{q_i}-t\right) - \frac{p_i-b_{i|1}}{p_i-k_i}\Psi_{i|j=1}\right]}^{M_i} - y_{i|j=1}; \overbrace{\vartheta^2_{i|j=1} + \sigma^2\left(t_{q_i}-t\right)}^{W_i}\right].$$

Finally, using the moment generating function of a log-normal distribution we arrive at:

$$\widetilde{E}\left[\frac{S\left(t_{q_i}\right)}{G_{i|j=1}}\right] = S(t)^{\frac{b_{i|1}-k_i}{p_i-k_i}} e^{\alpha M_i + \frac{W_i}{2} - y_{i|j=1}}.$$

Using this last result we arrive at:[32]

$$c_i = e^{-r_d\left(t_{q_i}-t\right) + \theta_{i|j=1} + \frac{1}{2}\vartheta^2_{i|j=1}}\left[1-H_i\right]\left[S(t)^{\frac{b_{i|1}-k_i}{p_i-k_i}} e^{\alpha M_i + \frac{W_i}{2} - y_{i|j=1}} N(d_{i,1}) - N(d_{i,2})\right] \qquad (4)$$

where $N(.)$ is the value of the cumulative density of a standard normal distribution, $d_{i,2} = d_{i,1} - \sqrt{W_i}$, and $d_{i,1} = \left[1/\sqrt{W_i}\right]\left[\frac{b_{i|1}-k_i}{p_i-k_i} lnS(t) + \alpha M_i + W_i - y_{i|j=1}\right]$.

As a result, from equation (4) in (1), the value of Ratchet call and put options auctioned by the

and $X_2 \sim \Lambda(\mu_2, \sigma_2^2)$, then $X_1 X_2 \sim \Lambda(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$.

[32] If $X \sim \Lambda(\mu, \sigma^2)$ then $E[max(X-K,0)] = E(X)N(d_1) - KN(d_2)$ where $d_1 = \frac{ln[E(x)/K] + \frac{\sigma^2}{2}}{\sigma}$ and $d_2 = \frac{ln[E(x)/K] - \frac{\sigma^2}{2}}{\sigma} = d_1 - \sigma$. Similarly $E[max(K-X,0)] = KN(-d_2) - E(X)N(-d_1)$ (Hull, 2015).

CBoC, are equal to:[33]

$$C = N^{-1}\sum_{i=1}^{n}\left\{e^{-r_d\left(t_{q_i}-t\right)+\theta_{i|j=1}+\frac{1}{2}\vartheta_{i|j=1}^{2}}\left[1-H_i\right]\left[S\left(t\right)^{\frac{b_{i|1}-k_i}{p_i-k_i}}e^{\alpha M_i+\frac{W_i}{2}-y_{i|j=1}}N\left(d_{i,1}\right)-N\left(d_{i,2}\right)\right]\right\} \quad (5)$$

$$P = N^{-1}\sum_{i=1}^{n}\left\{e^{-r_d\left(t_{q_i}-t\right)+\theta_{i|j=1}+\frac{1}{2}\vartheta_{i|j=1}^{2}}H_i\left[N\left(-d_{i,2}\right)-S(t)^{\frac{b_{i|1}-k_i}{p_i-k_i}}e^{\alpha M_i+\frac{W_i}{2}-y_{i|j=1}}N(-d_{i,1})\right]\right\} \quad (6)$$

In Appendix A we derive the Deltas, Gammas, and Vegas for these options.

6 Results

6.1 Our Model Results

Employing equations 5, 6, and the equations of the greeks derived in the Appendix A, we estimate with our dataset the historic value given by our model for the call and put options, and its greeks, every fifteen minutes for every trading day. In Figures 3 and 4 we see the estimated values with our model of the call and put options. In Figure 11 wee see that these values are mostly influenced by the market volatility, while in Figure 12 the connection between the FX rate and the value of these instruments is low (Table 2).

Table 2: Correlation coefficients between the prices, deltas, gammas and vegas with the FX rate and the market volatility

	Call P	Put P	Call Δ	Put Δ	Call Γ	Put Γ	Call ν	Put ν
FX rate	6.23%	1.07%	2.40%	4.11%	2.55%	−6.86%	10.60%	2.11%
Volatility	85.71%	85.08%	64.70%	47.15%	14.56%	4.11%	72.49%	64.11%

SOURCE: Authors' calculations. All the coefficients have significance with a confidence level of 99%.

The lack of relationship of these values with the FX rate is evidenced in Figures 13, 14, 15, and 16, in which we see the estimated deltas and gammas with our model and their relationship with the market volatility and the FX rate (Table 2).

Additionally, both the deltas and gammas have negligible values. On average, according to the deltas, each call and put option over one thousand dollars is equivalent to being on a long position in

[33] Note that for put options, $g = ln(1-f)$.

1.70 and 0.56 dollars respectively.[34] On the other hand, the strong influence of the market volatility in the value of these options is endorsed by Figures 17 and 18, in which wee see the relationship of the estimated vega with the market volatility and the FX rate (Table 2). On average, the vegas of call an put options over one thousand dollars are 192 and 155, respectively.

In theory, a central bank that intervenes with *plain vanilla* options can outsource FX balancing effects through the dynamic hedging from its counterparts by shorting call (put) options when the FX rate is depreciating (appreciating). The positive (negative) delta in call (put) options generates incentives for hedging the option risk by selling (buying) in the spot market whenever the rate rises (declines); these dynamics generate a balancing effect that partially offsets the pressure on the FX rate.

The resulting greeks estimated with our model allow us to conjecture that the mechanisms in the portfolio balancing channel through which the CBoC FX options generate their expected effects on the hedging strategies are not the same as in plain vanilla options. The bearer's exposure is almost entirely in volatility, and in order to dynamically cover this risk, *vega neutral* hedges are required. The most straightforward tactic for these financial intermediaries to accomplish this coverage is by shorting options on the FX market which, in the case of emerging market economies, are mostly demanded by firms from the real sector with the objective to cover currency mismatches. For the Colombian case, net exporters (importers) that mostly have their income (costs) denominated in foreign currency have an appreciation (depreciation) risk in which the costs of hedging are reduced by setting up portfolios with short positions in *out-of-the-money* call (put) and long positions in *near-the-money* put (call) (Lemus, 2017). If these firms set up their portfolios when it is financially sound, i.e. exporting (importing) firms set them up when there are depreciating (appreciating) pressures in the FX market, financial intermediaries will end up with positive (negative) delta exposure in the foreign exchange, and will sell (buy) in the spot market to hedge their exposure. As a consequence, financial intermediaries achieve vega neutral positions, and through their spot market operations outsource *leaning against the wind* interventions from the central bank.

[34] The positive values for some of the put deltas are explained by the fact that an increase on the foreign exchange rate facilitates on the following days the fulfillment of the exercise condition. For example for one hundred thousand simulations with twenty periods, $r_d = 10\%$, $r_f = 5\%$, $\sigma = 2\%$, and $f = 4\%$, we have that an increase on the foreign exchange rate of 5% during the tenth day increases the probability that the exercise condition for put options is satisfied from 5.6% to 6.8%.

6.2 Contending Valuation Results

Table 4 presents, for the historically auctioned call and put options described in Section 3, the minimum and maximum bid, the cutoff price, and the values given by the LSM simulations and our model. Looking at the predicted value of these options, we find that the prediction of the LSM algorithm is in the range of the minimum and maximum observed bids 63% of the time, while our stochastic model based on the WTV methodology falls within the same range in 66% of the auctions (Figures 19 and 20).

Table 3: Average percentage deviation of the predicted premium over the average bid

	All Options	Call Options	Put Options
LSM	-24%	-3%	-39%
WTV	31%	68%	2%

SOURCE: Authors' calculations.

Moreover, for all options the LSM predicted premium has an average percentage deviation over the average bid that is smaller in absolute terms than the one for the WTV. For call options the LSM methodology has a more precise prediction in call options, and in put options the WTV surpasses the prediction of the LSM (Table 3). We conjecture that the underestimation of the LSM algorithm is explained by the absolute unawareness of the intermediate periods, while the overestimation of our model, following Stozitzky (2015), is attributable to the low level of liquidity of these instruments that contradicts the assumption of no transaction costs in our BSM framework model. Lastly, we note that our stochastic model requires only 0.38% and 0.016% of the computational effort compared to the LSM algorithm, when considering one and ten thousand trajectories, respectively.

7 Conclusions

In this paper we develop a simple yet powerful new approach to approximate the value of American options through an extension of the Black and Scholes (1973) and Merton (1973) model. Specifically, we develop a methodology that replicates the value of an American option through a portfolio of European options in which the weight of each option is related to its time value. Our numerical exercises demonstrate that in the context of *plain vanilla* options, a well defined Weighted Time Value (WTV) portfolio gives values for American options that are comparable to the values predicted by the Least Squares Monte Carlo (LSM) methodology. Overall, we show that our WTV method outperforms LSM simulations in terms of mean absolute error (MAE) and weighted average absolute percentage error (WAPE). Additionally, our valuation exhibits less volatile valuation trajectories.

Following this idea, we deconstruct the different exotic features behind currency options employed by the Central Bank of Colombia (CBoC) during 1999-2012. In particular, these features comprise a combination of: i) Ratchet options, ii) Asian options with an average strike, and iii) options with stochastic barriers. We find that the LSM methodology, when applied to financial derivatives with exotic and multidimensional features, can produce estimations with numerical issues. Our methodology overcomes these problems and predicts premiums that are consistent with the bids observed in the auctions of the CBoC. Additionally, the bearers of these instruments are mainly exposed to FX volatility, and, in order to hedge this exposure, require *vega neutral* strategies. Moreover, when pricing options with exotic attributes, we find that our method requires less than 1% of the computational effort compared to that of the LSM.

We believe that our methodology can have useful implications for active practitioners that employ currency derivatives. This includes central banks in countries such as Mexico, Chile, Australia, and Colombia which have used FX options as an intervention mechanism. In some of these cases, exotic option features have been enacted. Our method can be extended and adjusted to a wide variety of option structures and allow central banks to evaluate (ex-ante) the expected option price and the channels through which dynamic hedging operates in complex instruments.

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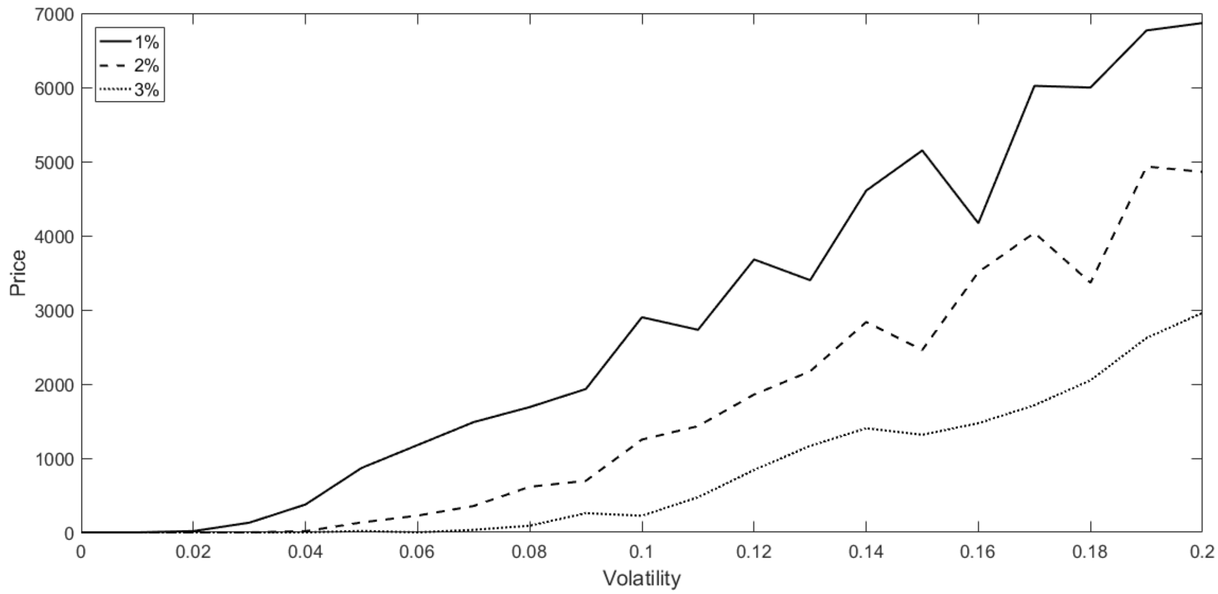
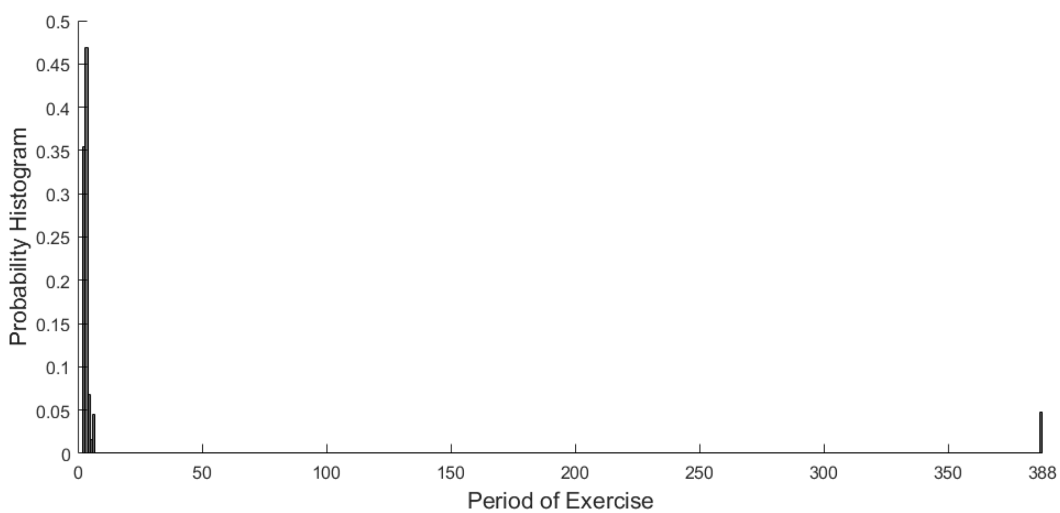
FIGURES

FIGURE 1. Call Price (LSM algorithm) for option issued on February 12, 2009



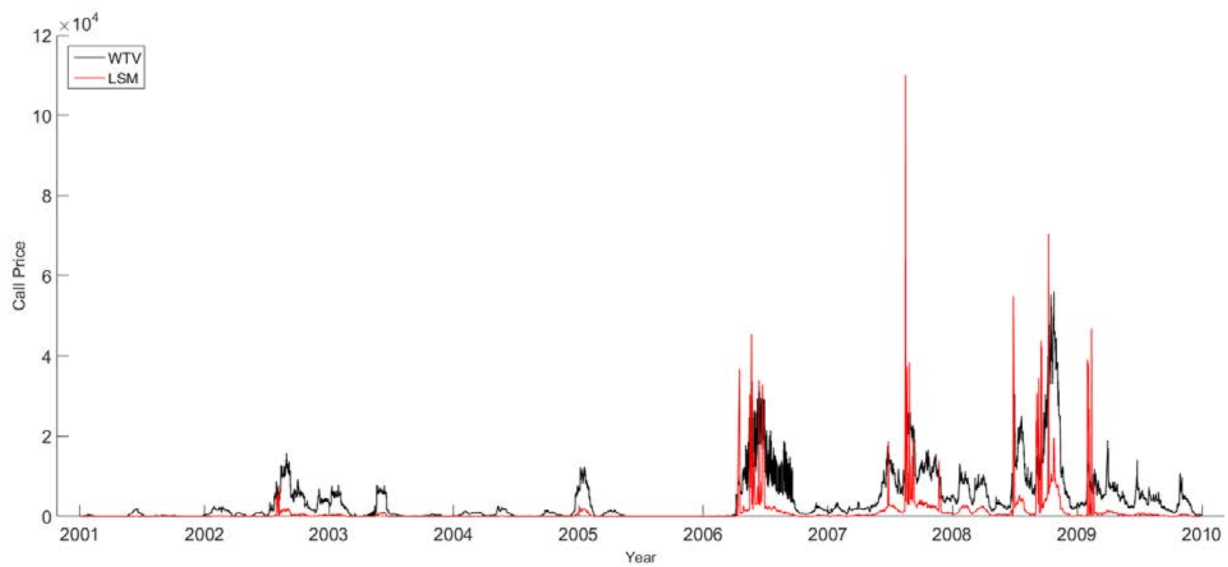
Price given by the LSM algorithm for the option auction by the CBoC on February 12th of 2009 with f equal to 1%, 2% and 3%. The domestic and foreign interest rate are respectively 9% and 3.25%. The value of the option is calculated at 11 am and the option lasts for 20 days.

FIGURE 2. Probability histogram of exercise periods with the LSM algorithm



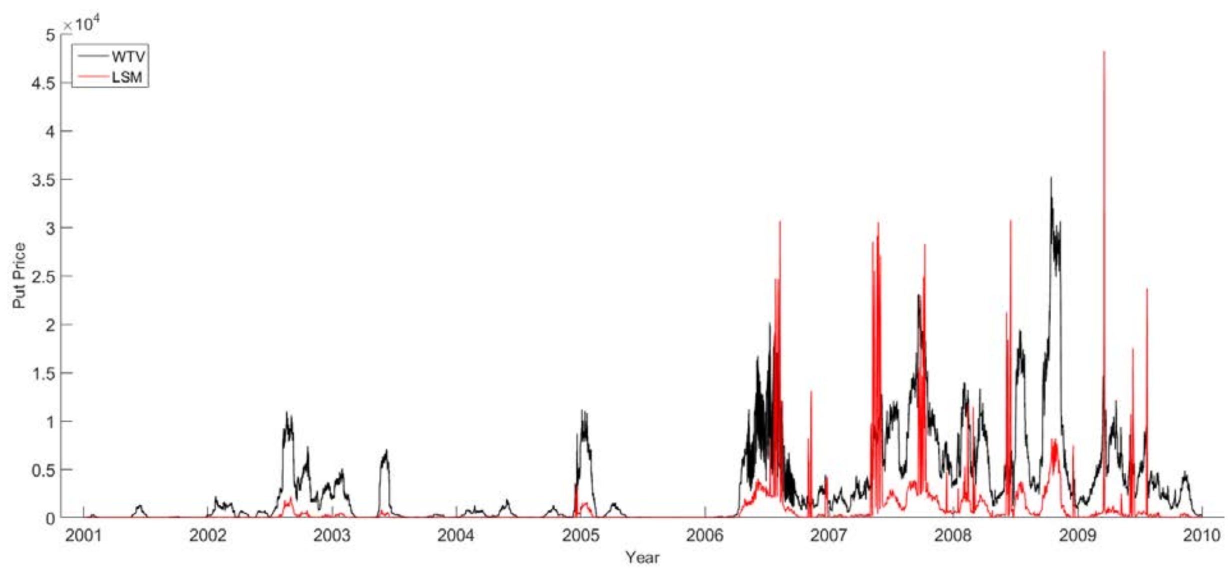
For each of the 17 call and 21 put options we estimate the optimal strategy period. The price of each one of the options is calculated using one thousand trajectories and the prevailing market conditions at the moment of the auction.

FIGURE 3. Volatility Call Price with WTV and LSM



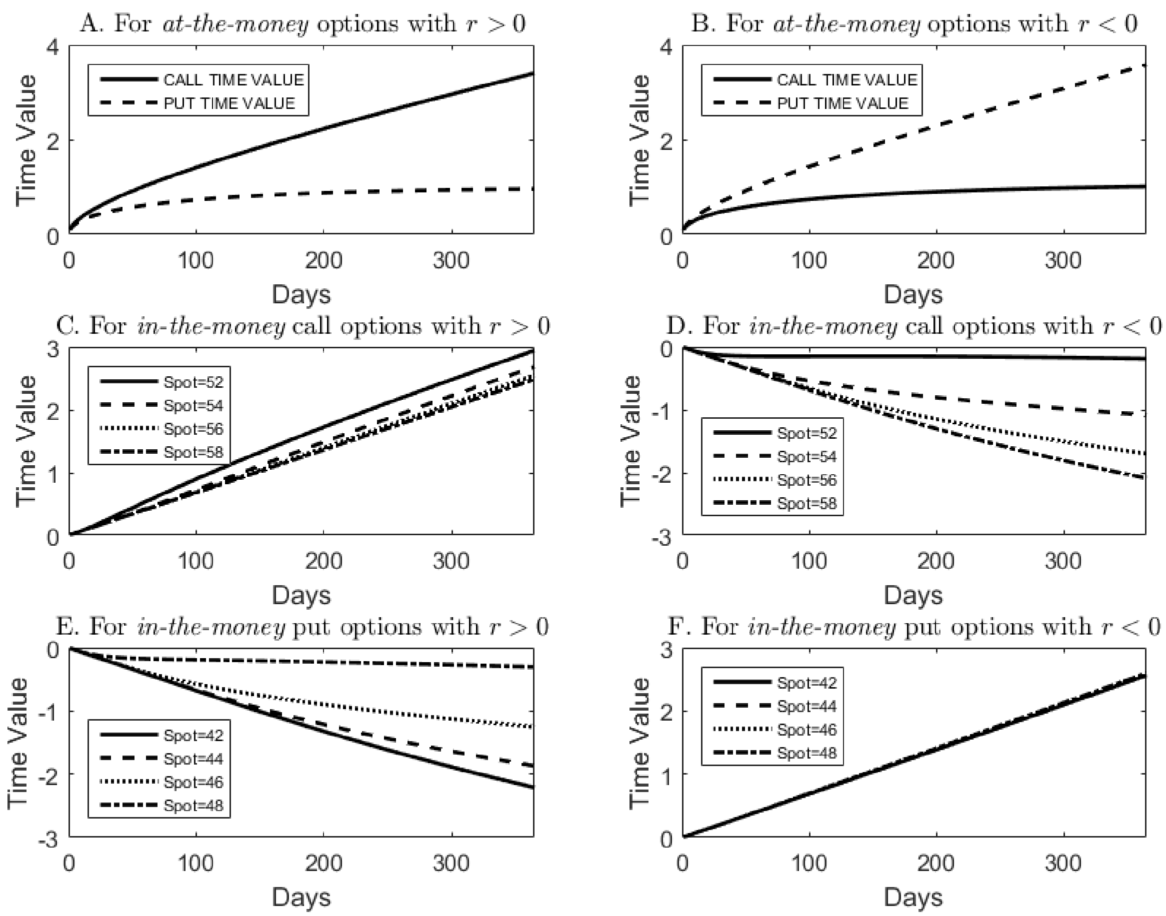
In both models we use the Central Bank of Colombia intervention rate as domestic rate, the prime rate as foreign rate, the market volatility of the last month and the assumption that the option will last for 20 days. The LSM value is given each day at 11 am, while the WTV is for every fifteen minutes on trading days and hours. The LSM simulations use one thousand trajectories and assume 20 possible exercises each day.

FIGURE 4. Volatility Put Price with WTV and LSM



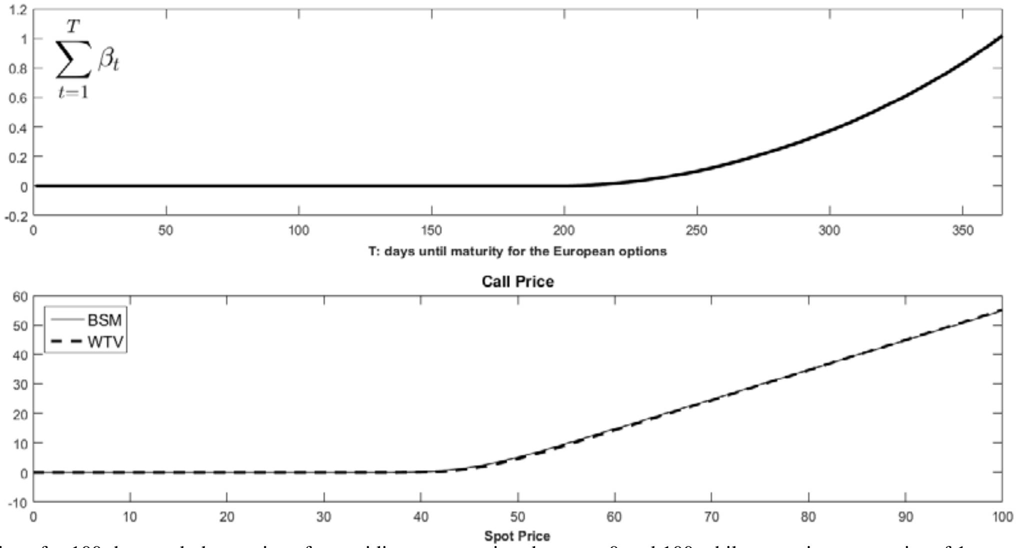
In both models we use the Central Bank of Colombia intervention rate as domestic rate, the prime rate as foreign rate, the market volatility of the last month and the assumption that the option will last for 20 days. The LSM value is given each day at 11 am, while the WTV is for every fifteen minutes on trading days and hours. The LSM simulations use one thousand trajectories and assume 20 possible exercises each day.

FIGURE 5. Time Value for call and put options with different spot prices



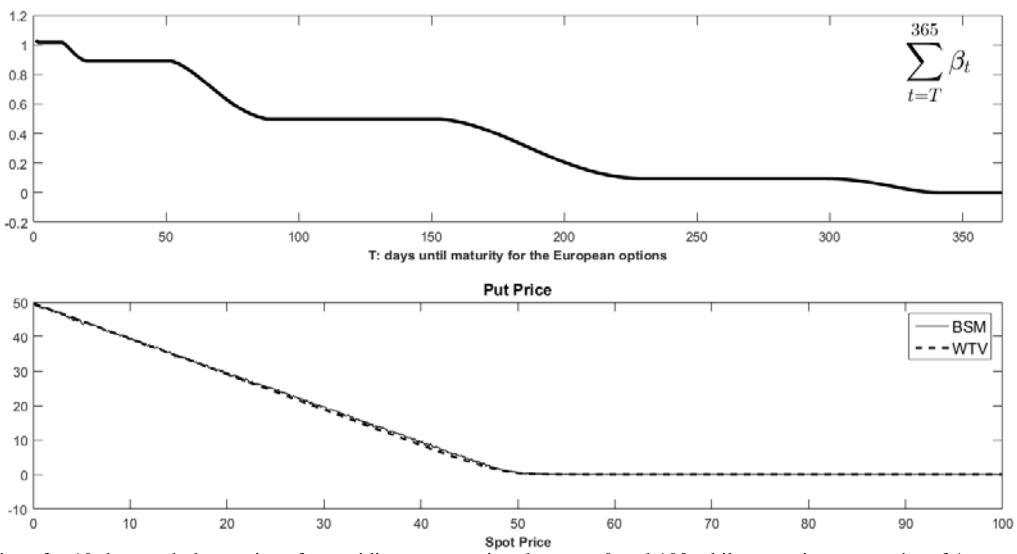
We assume an option with a strike of 50, an annualized volatility of 10% and an interest rate of 5% when positive and -5% when negative.

FIGURE 6. $g(t\upsilon_i)$ Functional Form in Call Options



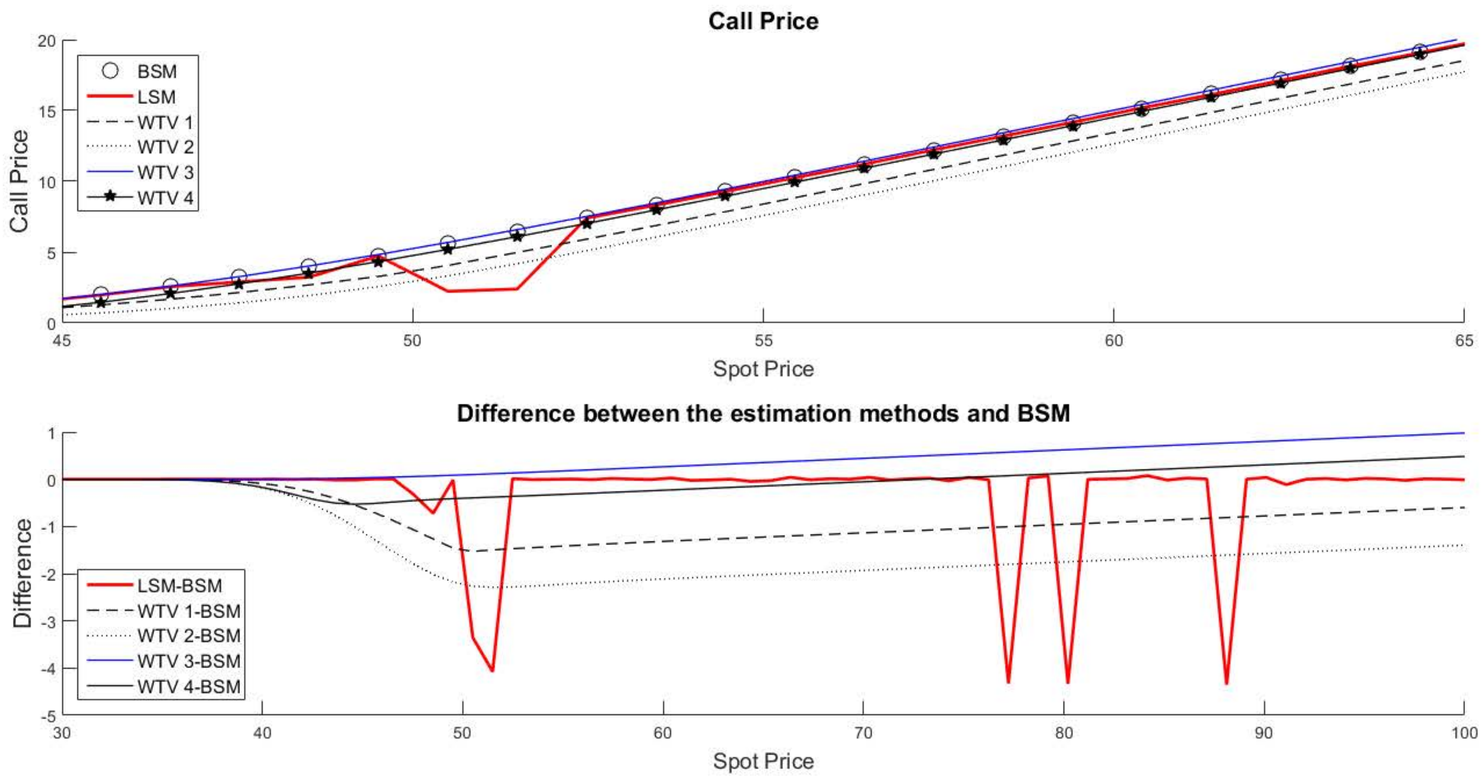
The regression is given for 100 thousand observations for equidistant spot prices between 0 and 100 while assuming a maturity of 1 year, a strike of 50, an interest rate of 10% and an annualized volatility of 10%. The WTV portfolio is composed of 365 options, each one with a different expiration day. The aggregated notional amount needed to minimize the least squares residuals is $m = 1.0179$.

FIGURE 7. $g(t\upsilon_i)$ Functional Form in Put Options



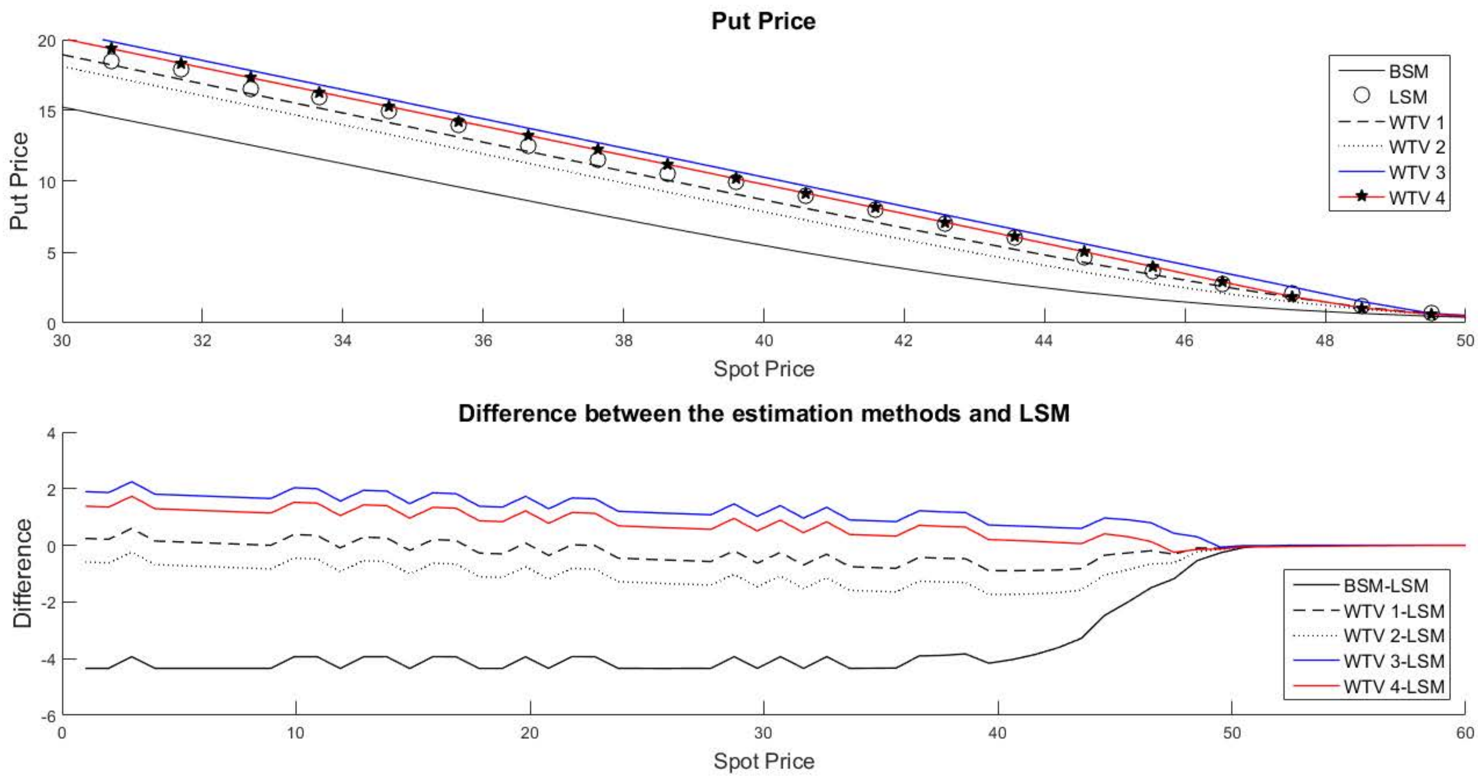
The regression is given for 10 thousand observations for equidistant spot prices between 0 and 100 while assuming a maturity of 1 year, a strike of 50, an interest rate of 10%, and an annualized volatility of 10%. In each of the simulations of the LSM we assume twelve exercises and ten thousand trajectories. The WTV portfolio is composed of 365 options, each one with a different expiration day. The aggregated notional amount needed to minimize the least squares residuals is $m = 1.0305$.

FIGURE 8. Call Price: BSM, LSM, and WTV



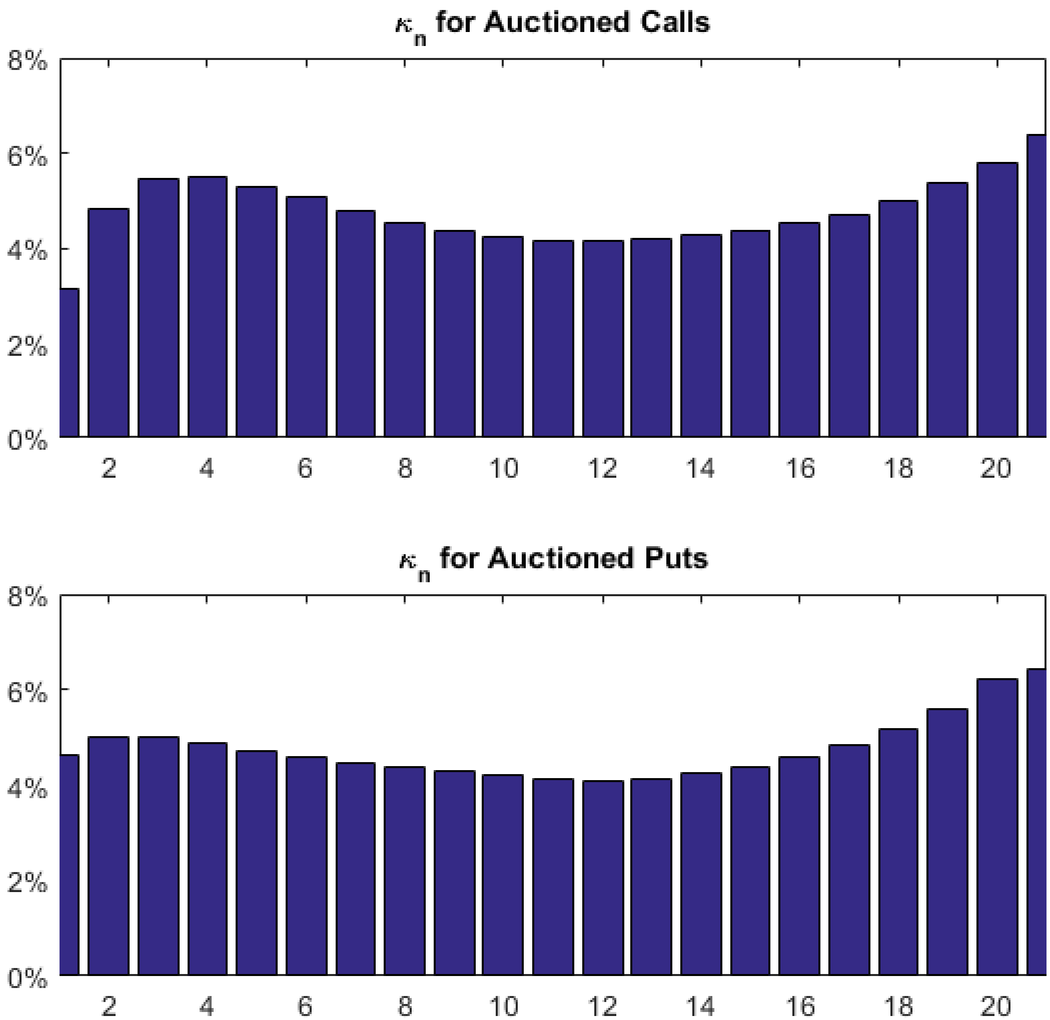
We assume a call option with a maturity of one year, a strike of 50, a domestic interest rate of 10% and an annualized volatility of 10%. For the LSM methodology we assume 12 possible exercises and one hundred thousand possible trajectories. For the WTV we assume 365 different options, each one with a different day of maturity. Particularly we look at the WTV given by: i) a linear weighting of the *time value* $(tv_i/\sum_{i=1}^{N} tv_i)$ (WTV 1); ii) a simple average of the value of the options in the portfolio (WTV 2); iii) the value given by allocating all the weight to the European option with the greatest *time value* (WTV 3); and iv) the value given by the allocation of weights according to the formula $exp(2tv_i)/\sum_{i=1}^{N} exp(2tv_i)$ (WTV 4). These WTV portfolios are built with the previously found aggregate notional amount for call optionS of $m = 1.0179$. In order to avoid negative weights (when there is a negative *time value*) we use the *time values* after netting out the minimum *time value* of the set of possible European options, this is $tv_i - Min(tv_i)$.

FIGURE 9. Put Price: BSM, LSM, and WTV



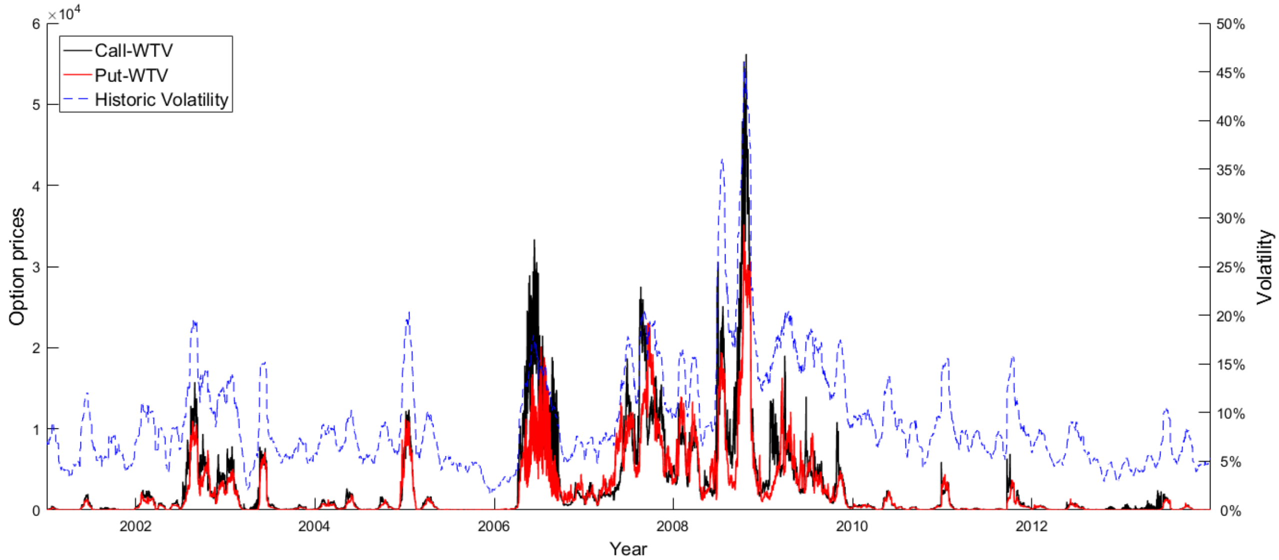
We assume a put option with a maturity of one year, a strike of 50, a domestic interest rate of 10% and an annualized volatility of 10%. For the LSM methodology we assume 12 possible exercises and one hundred thousand possible trajectories. For the WTV we assume 365 different options, each one with a different day of maturity. Particularly we look at the WTV given by: i) a linear weighting of the *time value* $(tv_i/\sum_{i=1}^{N} tv_i)$ (WTV 1); ii) a simple average of the value of the options in the portfolio (WTV 2); iii) the value given by allocating all the weight to the European option with the greatest *time value* (WTV 3); and iv) the value given by the allocation of weights according to the formula $exp(2tv_i)/\sum_{i=1}^{N} exp(2tv_i)$ (WTV 4). These portfolios are built with the previously found aggregate notional amount for put options of $m = 1.0305$. In order to avoid negative weights (when there is a negative *time value*) we use the *time values* after netting out the minimum *time value* of the set of possible European options, this is $tv_i - Min(tv_i)$.

FIGURE 10. κ_n for Auctioned Call and Put Options



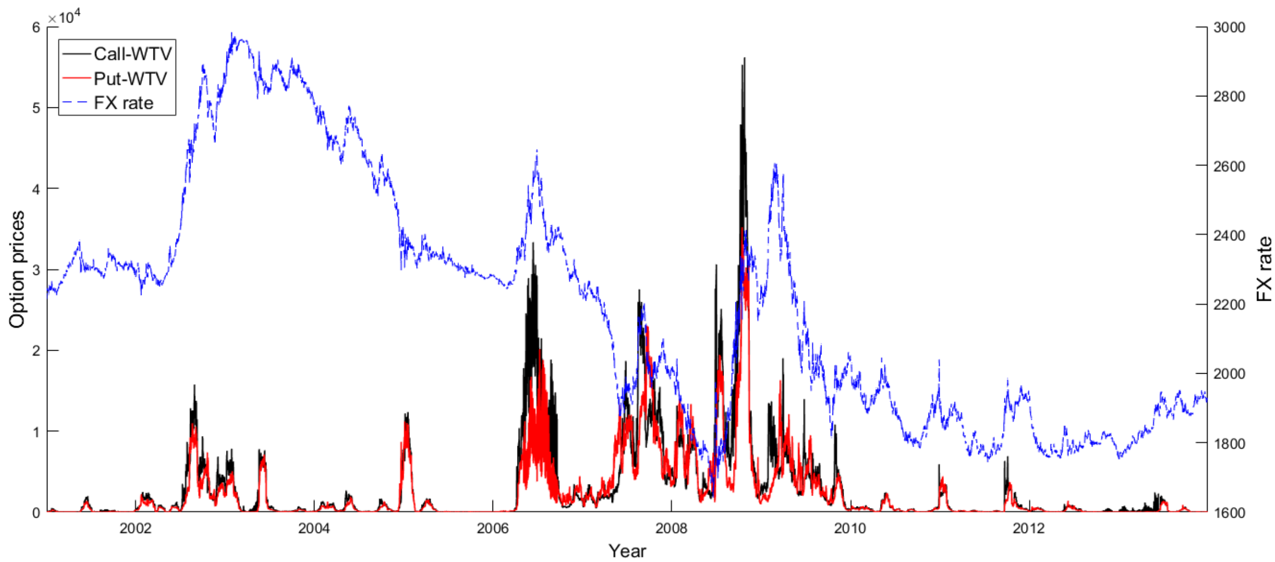
κ_n given for auctioned call and put options with a linear weighting of the *time value*. The *time value* is found by assuming the the current *intrinsic value* is applicable to all the European options in the portfolio.

FIGURE 11. Prices and Market Volatility



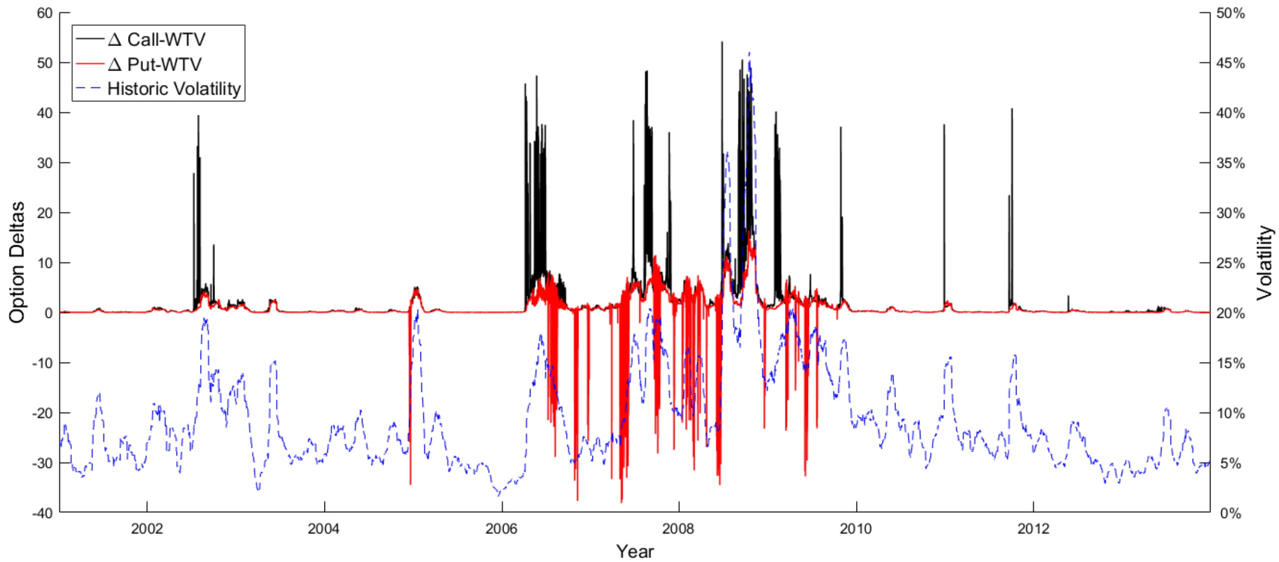
The WTV estimation uses the Central Bank of Colombia intervention rate as domestic rate, the prime rate as foreign rate, the market volatility of the last month and the assumption that the option will last for 20 days. The calculations are made every fifteen minutes on trading days and hours.

FIGURE 12. Prices and Spot FX rate



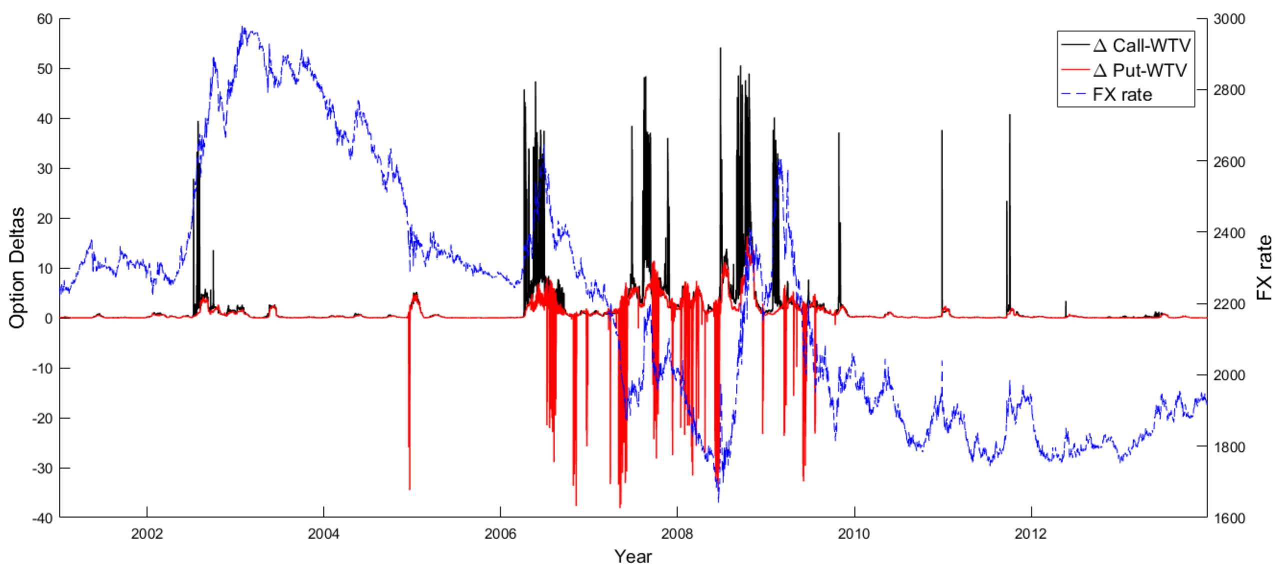
The WTV estimation uses the Central Bank of Colombia intervention rate as domestic rate, the prime rate as foreign rate, the market volatility of the last month and the assumption that the option will last for 20 days. The calculations are made every fifteen minutes on trading days and hours.

FIGURE 13. Deltas and Market Volatility



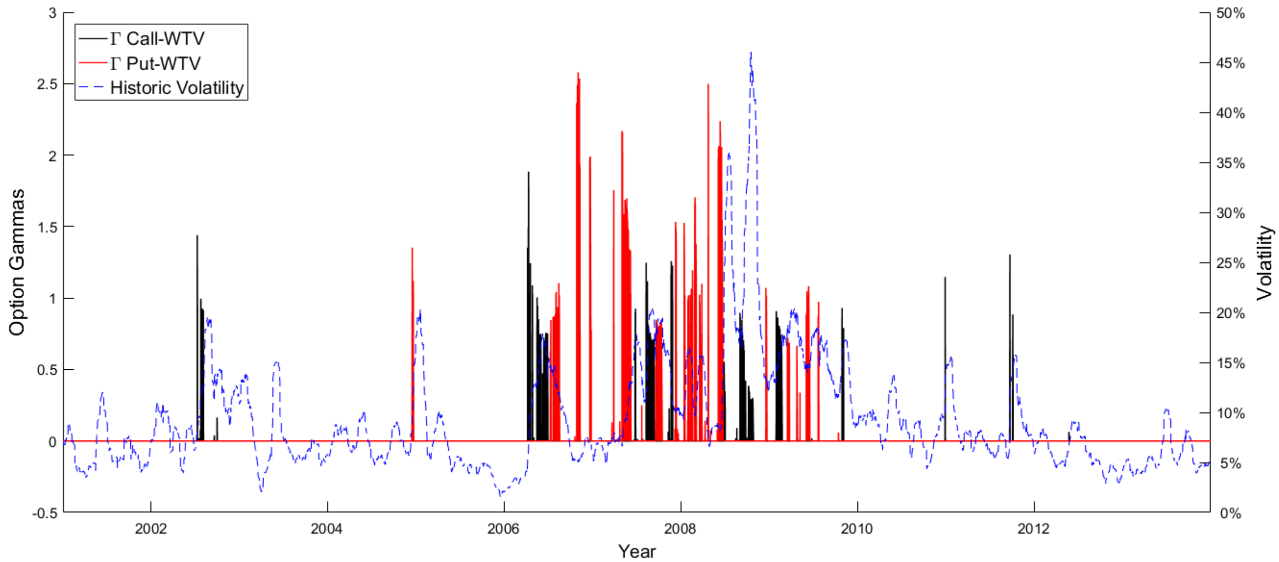
The WTV estimation uses the Central Bank of Colombia intervention rate as domestic rate, the prime rate as foreign rate, the market volatility of the last month and the assumption that the option will last for 20 days. The calculations are made every fifteen minutes on trading days and hours.

FIGURE 14. Deltas and Spot FX rate



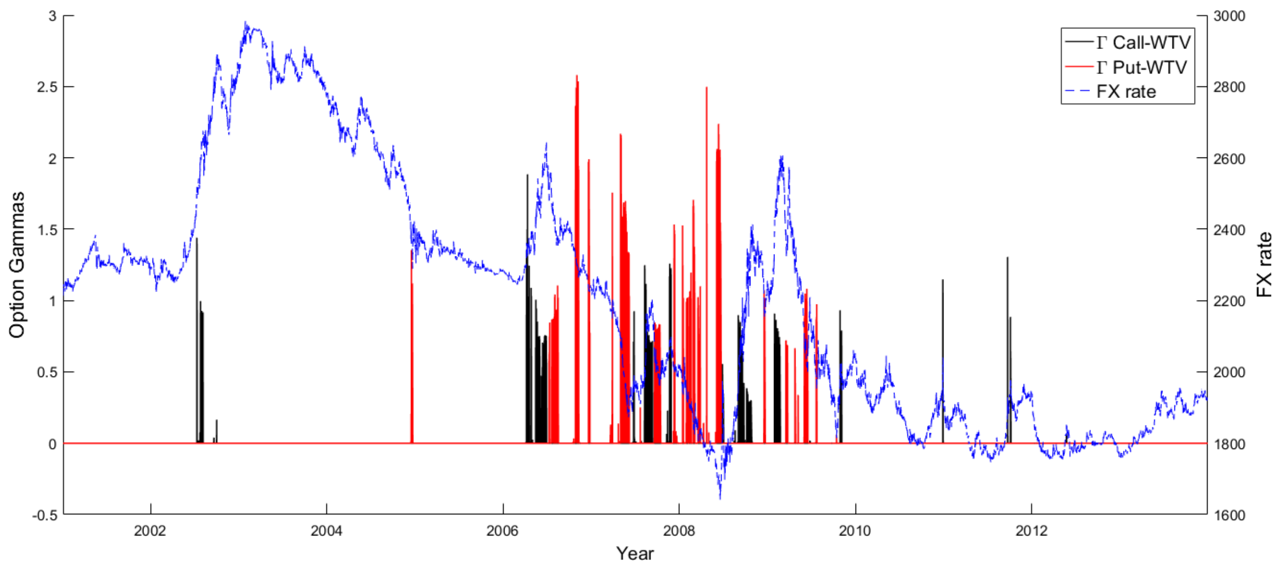
The WTV estimation uses the Central Bank of Colombia intervention rate as domestic rate, the prime rate as foreign rate, the market volatility of the last month and the assumption that the option will last for 20 days. The calculations are made every fifteen minutes on trading days and hours.

FIGURE 15. Gammas and Market Volatility



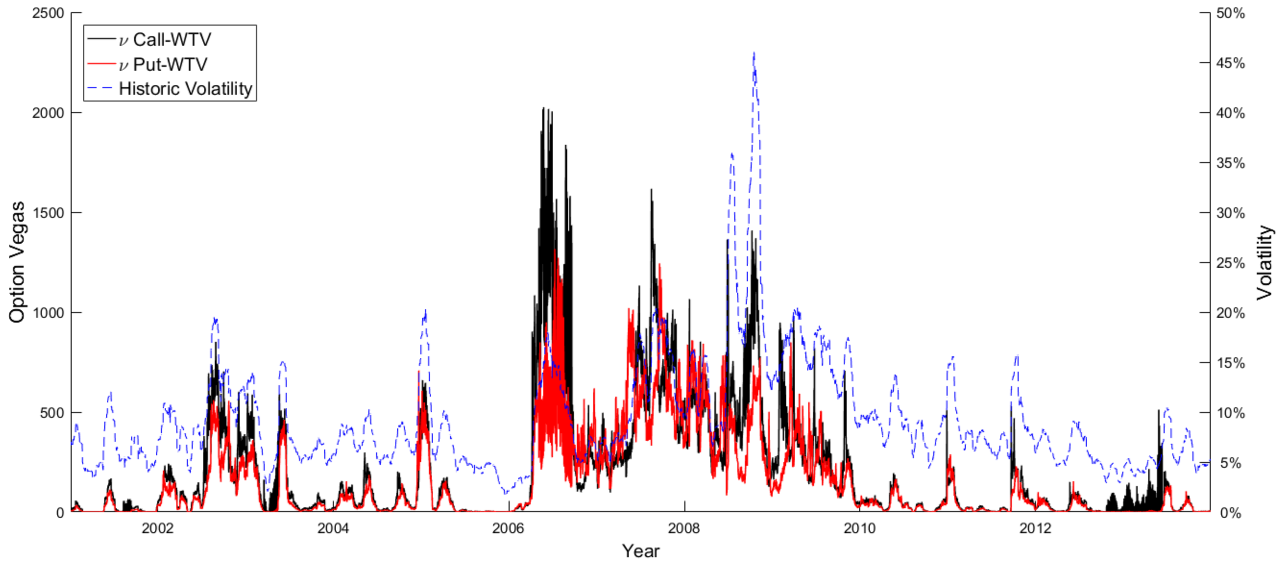
The WTV estimation uses the Central Bank of Colombia intervention rate as domestic rate, the prime rate as foreign rate, the market volatility of the last month and the assumption that the option will last for 20 days. The calculations are made every fifteen minutes on trading days and hours.

FIGURE 16. Gammas and Spot FX rate



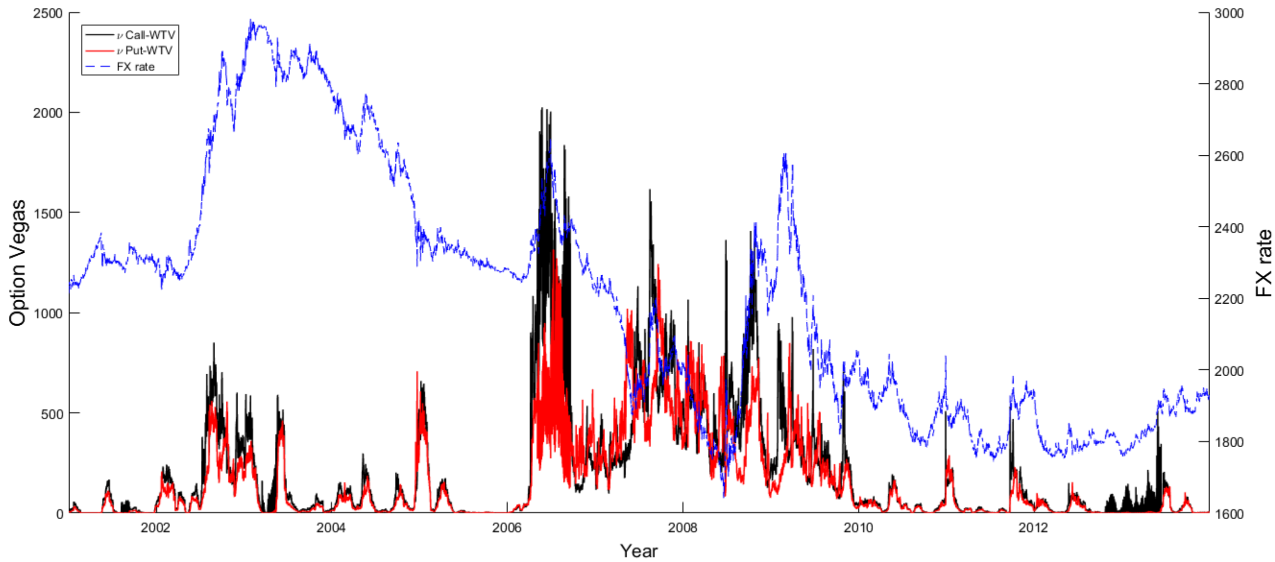
The WTV estimation uses the Central Bank of Colombia intervention rate as domestic rate, the prime rate as foreign rate, the market volatility of the last month and the assumption that the option will last for 20 days. The calculations are made every fifteen minutes on trading days and hours.

FIGURE 17. Vegas and Market Volatility



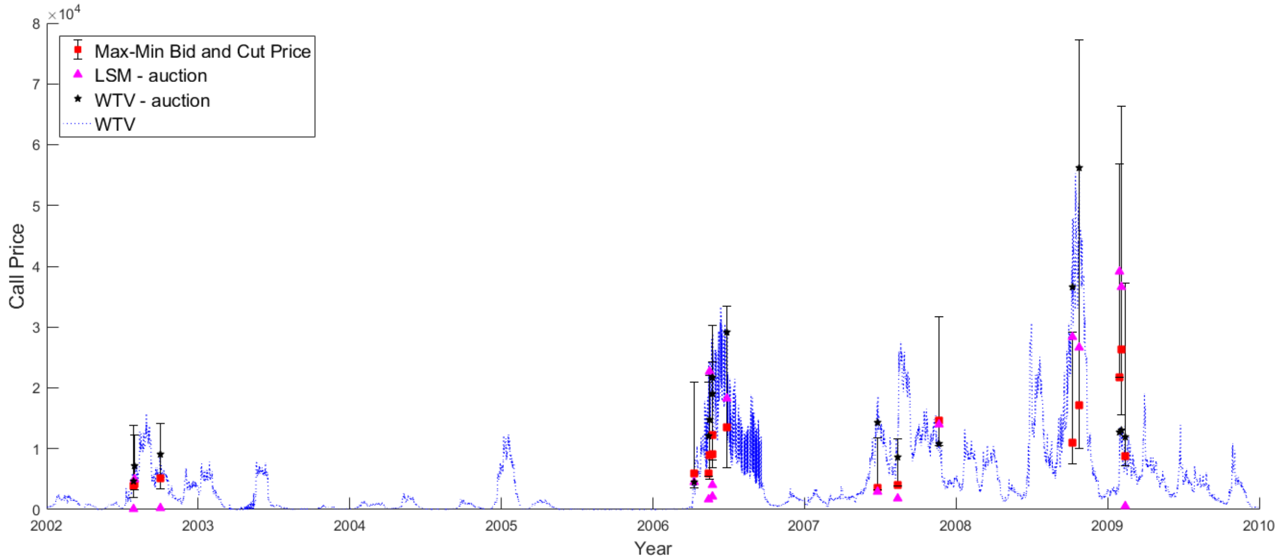
The WTV estimation uses the Central Bank of Colombia intervention rate as domestic rate, the prime rate as foreign rate, the market volatility of the last month and the assumption that the option will last for 20 days. The calculations are made every fifteen minutes on trading days and hours.

FIGURE 18. Vegas and Spot FX rate



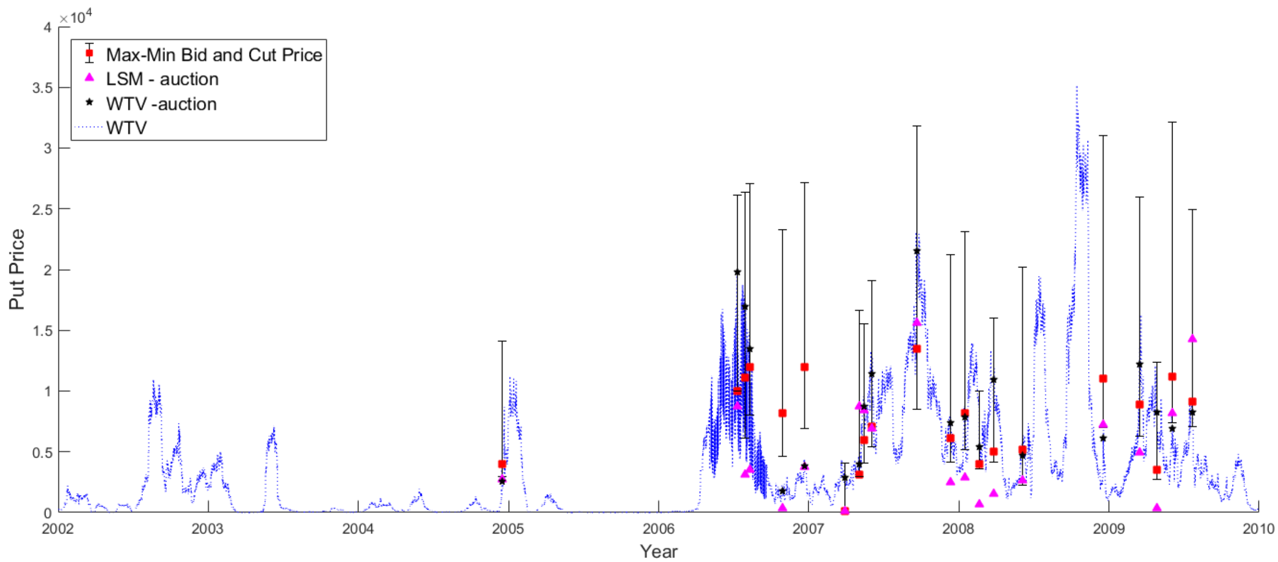
The WTV estimation uses the Central Bank of Colombia intervention rate as domestic rate, the prime rate as foreign rate, the market volatility of the last month and the assumption that the option will last for 20 days. The calculations are made every fifteen minutes on trading days and hours.

FIGURE 19. Volatility Calls: Maximum-Minimum Bids, cut prices, LSM value, and WTV value



The black ranges represent the maximum and minimum bids in each auction. The red square is the cutting prices of the action. The purple triangle is the corresponding LSM valuation and the black star the WTV estimation. The dotted line is the WTV price.

FIGURE 20. Volatility Puts: Maximum-Minimum Bids, cut prices, LSM value, and WTV value



The black ranges represent the maximum and minimum bids in each auction. The red square is the cutting prices of the action. The purple triangle is the corresponding LSM valuation and the black star the WTV estimation. The dotted line is the WTV price.

TABLE

Table 4: Results: Options issued by the Central Bank of Colombia: 1999-2012

	Date	Min Bid	Max Bid	Cut Price	Monte Carlo	Our Model
Call Options						
1	12/02/2009	1,500	28,500	8,700	614	9,128*
2	02/02/2009	10,900	40,000	26,400	36,597*	13,141*
3	30/01/2009	20.20	35,100	21,800	39,077	12,575*
4	24/10/2008	7,100	60,000	17,200	26,618*	48,088*
5	07/10/2008	3,500	18,100	11,000	28,407	38,237
6	22/11/2007	4,100	17,130	14,600	13,989*	10,871*
7	13/08/2007	100	7,670	4,000	1,835*	9,374
8	26/06/2007	250	8,200	3,500	2,935*	21,049
9	27/06/2006	6,625	20,000	13,501	18,252*	20,599
10	25/05/2006	5,250	18,000	12,200	2,141	18,609
11	23/05/2006	1,000	15,100	9,100	4,082*	19,231
12	18/05/2006	4,000	13,000	9,000	22,629	16,629
13	16/05/2006	500	15,000	6,000	1,732*	12,520*
14	10/04/2006	2,500	15,000	6,000	4,436*	3,961*
15	02/10/2002	1,800	9,010	5,157	304	7,956*
16	01/08/2002	1,000	8,010	4,220	5,014*	6,881*
17	29/07/2002	1,800	10,010	3,800	115	4,606*
Put Options						
1	22/07/2009	2000	15,850	9,100	14,289*	8,749*
2	03/06/2009	3750	21,000	11,150	8,211*	7,250*
3	27/04/2009	800	8,900	3,500	323	7,418*
4	17/03/2009	2600	17,100	8,900	4,928*	11,964*
5	18/12/2008	4025	20,000	11,050	7,211*	1,604
6	04/06/2008	2985	15,001	5,200	2,659	4,523*
7	25/03/2008	850	11,000	5,000	1,576*	9,206*
8	20/02/2008	400	6,020	4,001	689	4,824*
9	15/01/2008	3000	15,001	8,150	2,900	7,843*
10	11/12/2007	2001	15,100	6,130	2,488*	6,691*
11	20/09/2007	5000	18,300	13,500	15,581*	21,358
12	04/06/2007	1700	12,001	7,100	6,943*	12,143
13	15/05/2007	1900	9,570	6,000	8,396*	8,609*
14	03/05/2007	200	13,500	3,130	8,735*	3,873*
15	30/03/2007	100	4,000	100	128*	1,948*
16	21/12/2006	5100	15,125	12,000	3,770	3,761
17	30/10/2006	3500	15,100	8,150	330	1,333
18	10/08/2006	4000	15,100	12,000	3,529	12,758*
19	31/07/2006	5001	15,250	11,100	3,107	7,184*
20	11/07/2006	10	16,100	10,000	8,718*	18,711
21	17/12/2004	1001	10,110	4,000	2,696*	2,207*

SOURCE: Authors' calculations. In the models we use the Central Bank of Colombia intervention rate as domestic rate, the prime rate as foreign rate and the market volatility of the last month. In both models we calculate the value of the option at 11 am and assume the the option will last for 20 days. In the Monte Carlo simulation we use a 1000 paths and 20 possible exercise periods each day. * means that the value obtained is between the minimum and the maximum bids observed.

Appendix A: The Greeks

- By definition $N'(d_{i,j}) = \frac{1}{\sqrt{2\pi}} exp\left[-\frac{d_{i,j}^2}{2}\right]$. Given that $d_{i,1} = d_{i,2} + \sqrt{W_i}$ it can be shown that $N'(d_{i,1}) = N'(d_{i,2}) S(t)^{-\frac{b_{i|1}-k_i}{p_i-k_i}} e^{-\alpha M_i - \frac{W_i}{2} + y_{i|j=1}}$.
- $\frac{\partial d_{i,1}}{\partial S(t)} = \frac{\partial d_{i,2}}{\partial S(t)} = \frac{1}{\sqrt{W_i}}\left[\left(\frac{b_{i|1}-k_i}{p_i-k_i}\right)\frac{1}{S(t)} - \frac{\partial y_{i|j=1}}{\partial S(t)}\right]$.
- $\frac{\partial y_{i|j=1}}{\partial S(t)} = \left(\frac{1}{p_i-k_i}\right)\frac{1}{S(t)}$ if $t = b_{i|1}$, otherwise is equal to 0. Similarly $\frac{\partial^2 y_{i|j=1}}{\partial S(t)^2} = -\left(\frac{1}{p_i-k_i}\right)\frac{1}{S(t)^2}$ if $t = b_{i|1}$, otherwise is equal to 0.
- $\frac{\partial \theta_{i|j=1}}{\partial S(t)} = \frac{\partial y_{i|j=1}}{\partial S(t)} + \left(\frac{p_i - b_{i|1}}{p_i-k_i}\right)\frac{1}{S(t)}$ and $\frac{\partial \theta_{i|j=1}}{\partial \sigma} = -\sigma\left(\frac{p_i-b_{i|1}}{p_i-k_i}\right)\Psi_{i|j=1}$.
- $\frac{\partial d_{i,1}}{\partial \sigma} = \frac{\partial d_{i,2}}{\partial \sigma} + \frac{1}{2\sqrt{W_i}}\frac{\partial W_i}{\partial \sigma}$ and $\frac{\partial W_i}{\partial \sigma} = 2\sigma\left[\frac{p_1 - b_{i|1}}{(p_i-k_i)^2}\Upsilon_{i|j=1} + \left(t_{q_i} - t\right)\right]$.

With these results it can be proven that:

Deltas:

$$\Delta C = N^{-1}\sum_{i=1}^{N}\left\{e^{-r_d\left(t_{q_i}-t\right)+\theta_{i|j=1}+\frac{1}{2}\vartheta_{i|j=1}^2}\frac{[1-H_i]}{S(t)}\left[S(t)^{\frac{b_{i|1}-p_i}{p_i-k_i}} e^{\alpha M_i + \frac{W_i}{2} - y_{i|j=1}} N(d_{i,1}) - N(d_{i,2})\left(\frac{p_i - b_{i|1}}{p_i - k_i} + \frac{\partial y_{i|j=1}}{\partial S(t)} S(t)\right)\right]\right\} \quad (7)$$

$$\Delta P = N^{-1}\sum_{i=1}^{N}\left\{e^{-r_d\left(t_{q_i}-t\right)+\theta_{i|j=1}+\frac{1}{2}\vartheta_{i|j=1}^2}\frac{H_i}{S(t)}\left[N(-d_{i,2})\left(\frac{p_i - b_{i|1}}{p_i - k_i} + \frac{\partial y_{i|j=1}}{\partial S(t)} S(t)\right) - S(t)^{\frac{b_{i|1}-p_i}{p_i-k_i}} e^{\alpha M_i + \frac{W_i}{2} - y_{i|j=1}} N(-d_{i,1})\right]\right\} \quad (8)$$

Gammas:

$$\Gamma C = N^{-1}\sum_{i=1}^{N}\left\{e^{-r_d\left(t_{q_i}-t\right)+\theta_{i|j=1}+\frac{1}{2}\vartheta_{i|j=1}^2}\frac{[1-H_i]}{S(t)^2}\left[N'(d_{i,2})\frac{\partial d_{i,2}}{\partial S(t)} S(t)\left(\frac{b_{i|1}-k_i}{p_i-k_i} - \frac{\partial y_{i|j=1}}{\partial S(t)} S(t)\right) + N(d_{i,2})\left[\frac{\left(p_i - b_{i|1}\right)\left(b_{i|1}-k_i\right)}{(p_i-k_i)^2} - \frac{\partial^2 y_{i|j=1}}{\partial S(t)^2} S(t)^2 - \left(\frac{\partial y_{i|j=1}}{\partial S(t)} S(t)\right)^2 - 2\left(\frac{p_i - b_{i|1}}{p_i-k_i}\right)\frac{\partial y_{i|j=1}}{\partial S(t)} S(t)\right]\right]\right\} \quad (9)$$

$$\Gamma P = -N^{-1}\sum_{i=1}^{N}\left\{e^{-r_d\left(t_{q_i}-t\right)+\theta_{i|j=1}+\frac{1}{2}\vartheta_{i|j=1}^2}\frac{H_i}{S(t)^2}\left[N'\left(-d_{i,2}\right)\left(-\frac{\partial d_{i,2}}{\partial S(t)}\right)S(t)\left(\frac{b_{i|1}-k_i}{p_i-k_i}-\frac{\partial y_{i|j=1}}{\partial S(t)}S(t)\right)+\right.\right.$$

$$\left.\left.N\left(-d_{i,2}\right)\left[\frac{\left(p_i-b_{i|1}\right)\left(b_{i|1}-k_i\right)}{\left(p_i-k_i\right)^2}-\frac{\partial^2 y_{i|j=1}}{\partial S(t)^2}S(t)^2-\left(\frac{\partial y_{i|j=1}}{\partial S(t)}S(t)\right)^2-2\left(\frac{p_i-b_{i|1}}{p_i-k_i}\right)\frac{\partial y_{i|j=1}}{\partial S(t)}S(t)\right]\right]\right\} \quad (10)$$

Vegas:

$$\nu C = N^{-1}\sum_{i=1}^{N}\left\{e^{-r_d\left(t_{q_i}-t\right)+\theta_{i|j=1}+\frac{1}{2}\vartheta_{i|j=1}^2}\left[1-H_i\right]\sigma\left[2S(t)^{\frac{b_{i|1}-p_i}{p_i-k_i}}e^{\alpha M_i+\frac{W_i}{2}-y_{i|j=1}}N\left(d_{i,1}\right)\frac{p_i-b_{i|1}}{\left(p_i-k_i\right)^2}\Upsilon_{i|j=1}+\right.\right.$$

$$\left.\left.\frac{N'\left(d_{i,2}\right)}{\sqrt{(W_i)}}\left(\frac{p_i-b_{i|1}}{\left(p_i-k_i\right)^2}\Upsilon_{i|j=1}+\left(t_{q_i}-t\right)\right)-N\left(d_{i,2}\right)\left(\frac{p_i-b_{i|1}}{p_i-k_i}\right)\left(\frac{\Upsilon_{i|j=1}}{p_i-k_i}-\Psi_{i|j=1}\right)\right]\right\} \quad (11)$$

$$\nu P = N^{-1}\sum_{i=1}^{N}\left\{e^{-r_d\left(t_{q_i}-t\right)+\theta_{i|j=1}+\frac{1}{2}\vartheta_{i|j=1}^2}H_i\,\sigma\left[-2S(t)^{\frac{b_{i|1}-p_i}{p_i-k_i}}e^{\alpha M_i+\frac{W_i}{2}-y_{i|j=1}}N\left(-d_{i,1}\right)\frac{p_i-b_{i|1}}{\left(p_i-k_i\right)^2}\Upsilon_{i|j=1}+\right.\right.$$

$$\left.\left.\frac{N'\left(-d_{i,2}\right)}{\sqrt{(W_i)}}\left(\frac{p_i-b_{i|1}}{\left(p_i-k_i\right)^2}\Upsilon_{i|j=1}+\left(t_{q_i}-t\right)\right)+N\left(-d_{i,2}\right)\left(\frac{p_i-b_{i|1}}{p_i-k_i}\right)\left(\frac{\Upsilon_{i|j=1}}{p_i-k_i}-\Psi_{i|j=1}\right)\right]\right\} \quad (12)$$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